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# DISTORTIONS AND STRESSES OF PARABOLOIDAL SURFACE STRUCTURES PART I

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TABLE OF CONTENTS

NOMENCLATURE, SYMBOLS, AND ASSORTED MARKS

| | |
|---|---|
| $y^1, y^2, y^3$ | - rectangular Cartesian coordinates |
| $x^1, x^2, x^3$ | - general curvilinear coordinates |
| $\xi^1, \xi^2$ | - general coordinates on middle surface of undeformed shell |
| $\zeta$ | - coordinate normal to middle surface of undeformed shell |
| $\gamma, \theta$ | - polar parameters on middle surface of undeformed shell |
| $x, y$ | - cartesian parameters on middle surface of undeformed shell |
| $r$ | - radius of revolution of middle surface of undeformed paraboloid |
| $\theta$ | - angular coordinate of middle surface of undeformed paraboloid |
| $\gamma = \frac{r}{2f}$ | - slope of meridian tangent to middle surface of paraboloid |
| $\phi$ | - angle which meridianal tangent makes with tangent plane to apex of paraboloid (see figure 2.1.3) |
| $f$ | - focal length of middle surface of paraboloid |
| $\bar{r}_0$ | - radius vector to point on middle surface of undeformed shell |
| $\bar{r}$ | - radius vector to point in undeformed shell |

$\bar{R}_0$ - radius vector to point on middle surface of deformed shell

$\bar{R}$ - radius vector to point in deformed shell

$\bar{a}_1, \bar{a}_2$ - covariant base vectors of middle surface of undeformed shell

$\bar{a}^1, \bar{a}^2$ - contravariant base vectors of middle surface of undeformed shell

$\bar{n}$ - unit normal to middle surface of undeformed shell

$\bar{g}_1, \bar{g}_2, \bar{g}_3$ - covariant base vectors of undeformed body

$\bar{g}^1, \bar{g}^2, \bar{g}^3$ - contravariant base vectors of undeformed body

$g_{mn}$ - covariant metric tensor of undeformed body

$g^{mn}$ - contravariant metric tensor of undeformed body

$a_{mn}$ - covariant metric tensor of middle surface of undeformed shell

$a^{mn}$ - contravariant metric tensor of middle surface of undeformed shell

$\bar{G}_1, \bar{G}_2, \bar{G}_3$ - covariant base vectors of deformed body

$\bar{G}^1, \bar{G}^2, \bar{G}^3$ - contravariant base vectors of deformed body

$\bar{A}_1, \bar{A}_2$ - covariant base vectors of deformed middle surface of shell

| | |
|---|---|
| $\bar{A}^1, \bar{A}^2$ | - contravariant base vectors of deformed middle surface of shell |
| $\bar{N}$ | - unit normal vector to deformed middle surface |
| $R_1, R_2$ | - principal radii of curvature of middle surface of undeformed shell |
| $b_{11}, b_{12}, b_{22}$ | - second fundamental quadratic covariant tensor of undeformed middle surface of shell |
| $b_1^1, b_2^2, b_2^1, b_1^2$ | - second fundamental quadratic mixed tensor of undeformed middle surface of shell |
| $\gamma_{mn}, \gamma_n^m, \gamma^{mn}$ | - strain tensors |
| $\sigma^{mn}, \sigma_{mn}, \sigma_n^m$ | - stress tensors |
| $\ell^{\alpha\beta}$ | - stress resultant tensor of shell |
| $m^{\alpha\beta}$ | - moment resultant tensor of shell |
| $q^{\alpha}$ | - transverse shear resultant tensor of shell |
| $N_{\alpha\beta}$ | - physical components of force resultants tensors referred to $\xi^1, \xi^2$ coordinate system (units of force per unit length) |
| $M_{\alpha\beta}$ | - physical components of moment resultants tensors $\xi^1, \xi^2$ coordinate system (units of force-length per unit length) |
| $Q_{\alpha}$ | - physical components of transverse shear tensor $\xi^1, \xi^2$ co-ordinate system (units of force per unit length) |
| $\bar{i}_1, \bar{i}_2, \bar{i}_3$ | - unit base vectors associated with $y^1, y^2, y^3$ |

| | |
|---|---|
| $\epsilon_{\alpha\beta}$ | - covariant permutation surface tensor |
| $\epsilon^{\alpha\beta}$ | - contravariant permutation surface tensor |
| $N_r, N_\theta, N_{r\theta}$ | - force-resultants referred to $r, \theta$ coordinate system (units of force per unit length) |
| $M_r, M_\theta, M_{r\theta}$ | - moment-resultants referred to $r, \theta$ coordinate system (units of force-length per unit length) |
| $Q_r, Q_\theta$ | - transverse shear resultant referred to $r, \theta$ coordinate system (units of force per unit length) |
| $f$ | - focal length of parabola |
| $f^2, f^3, f^4, f^5$ | - powers of f, the focal length |
| $u^\sigma, u_\sigma$ | - displacement tensors of middle surface |
| $w^\sigma, w_\sigma$ | - rotation tensors of middle surface |
| $w$ | - displacement of middle surface along $\zeta$ (units of length) |
| $u^o_r$ | - displacement of middle surface along tangent to meridian (units of length) |
| $u^o_\theta$ | - displacement of middle surface along tangent to latitude (units of length) |
| $\gamma^o_{\alpha\beta}$ | - strain tensor of middle surface |
| $k_{\alpha\beta}$ | - strain-curvature tensor of middle surface |

| | |
|---|---|
| $\epsilon_r$ | - extensional strain along meridian (dimensionless) |
| $\epsilon_\theta$ | - extensional strain along latitude (dimensionless) |
| $\epsilon_{r\theta}$ | - shear strain (dimensionless) |
| $\epsilon_r^o$ | - extensional strain of middle surface along meridian (dimensionless) |
| $\epsilon_\theta^o$ | - extensional strain of middle surface along latitude (dimensionless) |
| $\epsilon_{r\theta}^o$ | - shear strain of middle surface (dimensionless) |
| $K_r$ | - extensional strain-curvature of middle surface along meridian (units of $(\text{length})^{-1}$) |
| $K_\theta$ | - extensional strain-curvature of middle surface along latitude ( units $(\text{length})^{-1}$ ) |
| $K_{r\theta}$ | - shear strain-curvature of middle surface (units of $(\text{length})^{-1}$ ) |
| $h$ | - thickness of undeformed shell (units of length) |
| $E$ | - Youngs modulus (units of force per unit area) |
| $\mu = \frac{E}{2(1+\nu)}$ | - shear modulus (units of force per unit area) |
| $\nu$ | - Poissons ratio (dimensionless) |
| $\rho_o$ | - weight-density (units of force per unit volume) |

| | |
|---|---|
| $\nu^2$ | - powers of $\nu$ , Poissons ratio |
| $\zeta^*$ | - coordinate normal to middle surface of deformed shell |
| $h^*$ | - thickness of deformed shell (units of length) |
| $\bar{\ell}^1, \bar{\ell}^2$ | - force-resultant vectors (units of force per unit length) |
| $\bar{m}^1, \bar{m}^2$ | - moment-resultant vectors (units of force-length per unit length) |
| $F^\alpha, F^3$ | - tensor components of body force vector |
| $\bar{P}$ | - body force vector (units of force per unit area) |
| $p_r, p_\theta, p_n$ | - physical components of body force vector (units of force per unit area) |
| $N_x, N_y, N_{xy}$ | - force-resultants referred to cartesian parameters (units of force per unit length) |
| $M_x, M_y, M_{xy}$ | - moment resultants referred to cartesian parameters (units of force-length per unit length) |
| $Q_x, Q_y$ | - transverse shear resultant referred to cartesian parameters (units of force per unit length) |
| $\gamma_1, \gamma_2$ | - bounding latitudes of paraboloidal shell |
| $(\bar{\ })$ | - signifies quantity under bar is a vector |
| $\|_\alpha$ or $\|_m$ | - vertical slash before subscript signifies covariant differentiation with respect to metric of deformed body |

$,_\alpha$ or $,_m$ - comma before subscript signifies covariant differentiation with respect to metric of undeformed body

$\left\{ {m \atop np} \right\}_0$ - Christoffel symbols of second kind of undeformed body

$\left\{ {\alpha \atop \beta\gamma} \right\}_0$ - Christoffel symbols of second kind of the middle surface of undeformed body

$\left\{ {m \atop np} \right\}$ - Christoffel symbols of second kind of deformed body

$\left\{ {\alpha \atop \beta\gamma} \right\}$ - Christoffel symbols of second kind of the middle surface of deformed body

$\bullet$ - dot between two vectors signifies scalar product

$\times$ - cross between two vectors signifies vector product

$B_{\alpha\beta}, B_\beta^\sigma$ - second fundamental tensors of deformed middle surface

$b_{\alpha\beta}, b_\beta^\sigma$ - second fundamental tensors of undeformed middle surface

# I Introduction

The quest for more and more precise radars in the ultra-high frequency regime has imposed very stringent requirements on the structural behavior of large antennae. Thus, the permissible deviations for an antenna surface which is to operate at 10,000 m.c. from a true paraboloidal surface of revolution is now felt to be 1/16 of the wavelength, which is 3/16 of a centimeter or .074 inches. Such a miniscule tolerance on the distortion of a structure which is a hundred feet or more in overall size, and furthermore, which is to assume different orientations with respect to the axis of gravity, requires an extremely high degree of sophistication in analysis, design, and construction.

The usual structures such as bridges, buildings, or even flight vehicles are designed mainly by strength considerations, although flight vehicles must also have a certain minimum stiffness in order to avoid aeroelastic difficulties. Machine tools are required to possess great stiffness, but machine tools are generally compact and weight limitations are practically non-existent. On the other hand, the primary design requirement of a high performance antenna is that the reflecting surface remain paraboloidal and, in the case of an antenna which is housed in a radome, strength considerations play a minor role in the design. Thus, the antenna must have great structural stiffness but since the main loads are its own dead weight, the structural stiffness must be accompanied by minimum weight, i.e., the antenna must possess a large ratio of structural stiffness to weight.

The basic structural components of the antennaare paraboloidal surface panels which, when joined together, form a surface of revolution. Such a structural configuration is generally called a "shell" although "surface structure" may be more appropriate. Our objective in this report is to treat in an exhaustive fashion the distortions and stresses in paraboloidal surface structures. We must, in view of the aforementioned stringent tolerances on the maintenance of the proper shape, investigate effects which generally can be ignored in the more common-place structural theory. The approach, in this report, will be to first lay the foundation for the general behavior of a paraboloidal shell. Then the equations will be specialized and simplified to the various forms of shell behavior which are classified as membrane behavior, etc. This report, which in a certain sense will never be completed, will be issued in sections since it is felt the best interests of the Lincoln Laboratory will be served in this manner rather than to delay publication until, say, 90% is completed.

In view of our desire to lay a general foundation, and to treat in an exhaustive fashion the behavior of paraboloidal surface structures, the authors feel that the pertinent equations and the geometry of the deformed structure can best be handled by the tensor calculus. In the more simple aspects such as membrane behavior with orthogonal shell coordinates, the advantages of the tensor calculus are minor and its use may even seem like the use of the theory of relativity to prove that a pitched baseball can curve. However, the power of the tensor calculus will become apparent as the more complex forms of surface structure behavior are considered.

## 2.1 GEOMETRY OF THE MIDDLE SURFACE: POLAR PARAMETERS

The middle surface of the paraboloidal shell of revolution is a surface which is generated by revolving the parabola

$$y^3 = \frac{(r)^2}{4f} \qquad 2.1.1$$

about the $y^3$ axis (see figure 2.1.1)

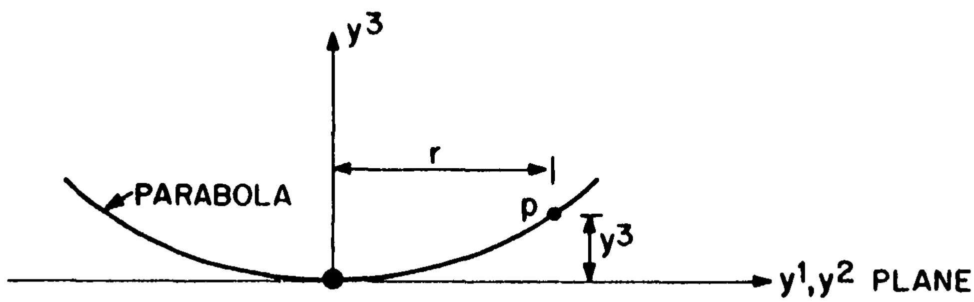


Figure 2.1.1

In this figure, r is the radius of revolution to a point p on the middle surface and f is the focal length of the parabola.

We will use $\bar{r}_0$ to denote the position vector from the origin of the rectangular cartesian axes to a point p on the surface of revolution (see figure 2.1.2). In terms of its rectangular components,

$\bar{r}_0$ is written as

$$\bar{r}_0(r,\theta) = r\cos\theta\,\bar{i}_1 + r\sin\theta\,\bar{i}_2 + \frac{(r)^2}{4f}\,\bar{i}_3$$

2.1.2

where the $\bar{i}_n$ are the unit vectors associated with the rectangular cartesian coordinates $y^n$, and $\theta$ is the angular coordinate measured in the $y^1 y^2$ plane from the $y^1$ axis.

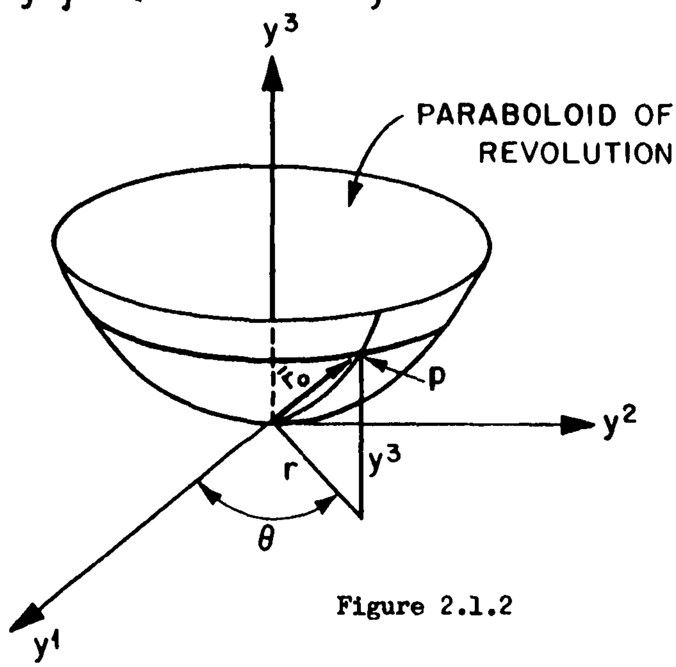


Figure 2.1.2

It is convenient at this stage to introduce a non-dimensional coordinate in place of r. Let

$$\gamma = \frac{r}{2f}$$

2.1.3

It should be recognized that $\gamma$ is the slope $\frac{dy^3}{dr}$ of the paraboloid and also that if $\phi$ is the angle to the tangent (see figure 2.1.3) then

$$\sin\phi = \frac{\gamma}{\sqrt{1+(\gamma)^2}} , \qquad 2.1.4$$

$$\cos\phi = \frac{1}{\sqrt{1+(\gamma)^2}} , \qquad 2.1.5$$

$$\tan\phi = \gamma , \qquad 2.1.6$$

$$\bar{r}_0 = 2f\left\{\gamma\cos\theta\,\bar{i}_1 + \gamma\sin\theta\,\bar{i}_2 + \frac{(\gamma)^2}{2}\,\bar{i}_3\right\} . \qquad 2.1.7$$

In this notation, the base vectors of the middle surface (see figure 2.1.3) are

$$\bar{a}_1 = \frac{\partial\bar{r}_0}{\partial\gamma} = 2f\left\{\cos\theta\,\bar{i}_1 + \sin\theta\,\bar{i}_2 + \gamma\,\bar{i}_3\right\} , \qquad 2.1.8$$

$$\bar{a}_2 = \frac{\partial \bar{r}_0}{\partial \theta} = 2f\left\{-\gamma \sin\theta\, \bar{i}_1 + \gamma \cos\theta\, \bar{i}_2\right\} . \qquad 2.1.9$$

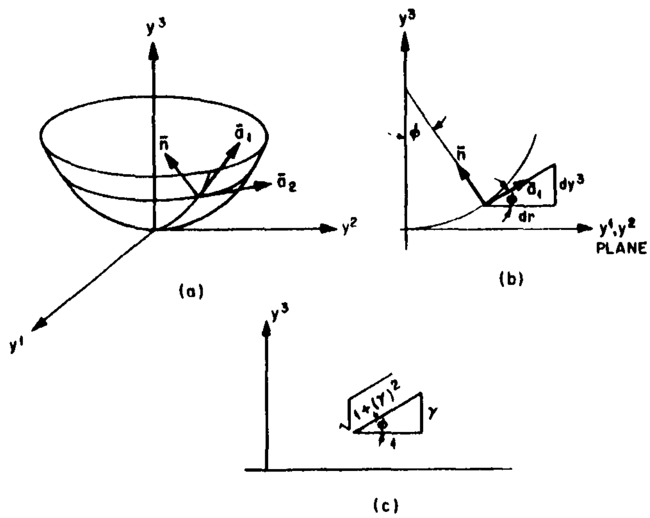


Figure 2.1.3

We summarize without detailed comment the pertinent geometrical parameters of the paraboloid.

The fundamental metric tensor

$$a_{\alpha\beta} = \bar{a}_\alpha \cdot \bar{a}_\beta \qquad 2.1.10$$

has the components

$$\left[a_{\alpha\beta}\right] = \begin{bmatrix} 4(f)^2\left[1+(\gamma)^2\right] & 0 \\ 0 & 4(f)^2(\gamma)^2 \end{bmatrix} \qquad 2.1.11$$

and the determinant, $a$, of the matrix $\left[a_{\alpha\beta}\right]$ is

$$a = 16(f)^4(\gamma)^2\left[1+(\gamma)^2\right]. \qquad 2.1.12$$

Since $a_{12} = 0$ the coordinate curves are orthogonal.

The unit normal, $\bar{n}$, to the surface

$$\bar{n} = \frac{1}{\sqrt{a}}\,\bar{a}_1 \times \bar{a}_2 \qquad 2.1.13$$

has the components

$$\bar{n} = -\frac{\gamma}{\sqrt{1+(\gamma)^2}}\cos\theta\,\bar{i}_1 - \frac{\gamma}{\sqrt{1+(\gamma)^2}}\sin\theta\,\bar{i}_2 + \frac{1}{\sqrt{1+(\gamma)^2}}\,\bar{i}_3 \qquad 2.1.14a$$

or in an alternative form,

$$\bar{n} = -\sin\phi\cos\theta\,\bar{i}_1 - \sin\phi\sin\theta\,\bar{i}_2 + \cos\phi\,\bar{i}_3\,. \qquad 2.1.14b$$

Note that the unit normal is directed inwardly (see figure 2.1.3).

We will need the derivatives of the base vectors.

$$\frac{\partial \bar{a}_1}{\partial \gamma} = 2f\,\bar{i}_3 \quad , \tag{2.1.15}$$

$$\frac{\partial \bar{a}_1}{\partial \theta} = \frac{\partial \bar{a}_2}{\partial \gamma} = 2f\left\{-\sin\theta\,\bar{i}_1 + \cos\theta\,\bar{i}_2\right\} , \tag{2.1.16}$$

$$\frac{\partial \bar{a}_2}{\partial \theta} = 2f\left\{-\gamma\cos\theta\,\bar{i}_1 - \gamma\sin\theta\,\bar{i}_2\right\} . \tag{2.1.17}$$

The second fundamental tensor of the paraboloidal surface,

$$b_{\alpha\beta} = \bar{n} \cdot \frac{\partial \bar{a}_\alpha}{\partial \gamma^\beta} \tag{2.1.18}$$

has the components

$$\left[b_{\alpha\beta}\right] = \begin{bmatrix} \dfrac{2f}{\sqrt{1+(\gamma)^2}} & 0 \\ 0 & \dfrac{2f\,(\gamma)^2}{\sqrt{1+(\gamma)^2}} \end{bmatrix} , \tag{2.1.19a}$$

$$[b_{\alpha\beta}] = \begin{bmatrix} 2f\cos\phi & 0 \\ 0 & 2f\gamma\sin\phi \end{bmatrix} . \qquad 2.1.19b$$

Since both $a_{12}$ and $b_{12}$ are zero, the coordinate curves $\gamma$ and $\theta$ are the lines of curvature of the paraboloidal surface of revolution.

The contravariant metric tensor which is defined by

$$a^{\alpha\beta} a_{\beta\gamma} = \delta^{\alpha}_{\gamma} \qquad 2.1.20$$

has the components

$$a^{11} = \frac{a_{22}}{a} = \frac{1}{a_{11}} = \frac{1}{4(f)^2\left[1+(\gamma)^2\right]} , \qquad 2.1.21$$

$$a^{12} = -\frac{a_{12}}{a} = 0 , \qquad 2.1.22$$

$$a^{22} = \frac{a_{11}}{a} = \frac{1}{a_{22}} = \frac{1}{4(f)^2(\gamma)^2} . \qquad 2.1.23$$

The results shown as $\frac{1}{a_{11}}$, $0$ ,and $\frac{1}{a_{22}}$ are a consequence of the orthogonality of the coordinate curves.

We will also have occasion to use the contravariant base vectors. These are

$$\bar{a}^1 = a^{1\beta}\,\bar{a}_\beta = a^{11}\,\bar{a}_1 = \frac{1}{a_{11}}\,\bar{a}_1 \quad , \qquad 2.1.24$$

$$\bar{a}^2 = a^{2\beta}\,\bar{a}_\beta = a^{22}\,\bar{a}_2 = \frac{1}{a_{22}}\,\bar{a}_2 \; . \qquad 2.1.25$$

The normal curvatures in the directions of the coordinate curves are also the principal normal curvatures because the coordinates are the lines of curvature. We will use $R_1$ and $R_2$ to denote the principal radii of curvature.

$$R_1 = \frac{a_{11}}{b_{11}} = 2f\left[1+(\gamma)^2\right]^{\frac{3}{2}} \quad , \qquad 2.1.26$$

$$R_2 = \frac{a_{22}}{b_{22}} = 2f\sqrt{1+(\gamma)^2} \; . \qquad 2.1.27$$

With reference to the principal radii of curvature, we can make the geometrical constructions shown in figure 2.1.4.

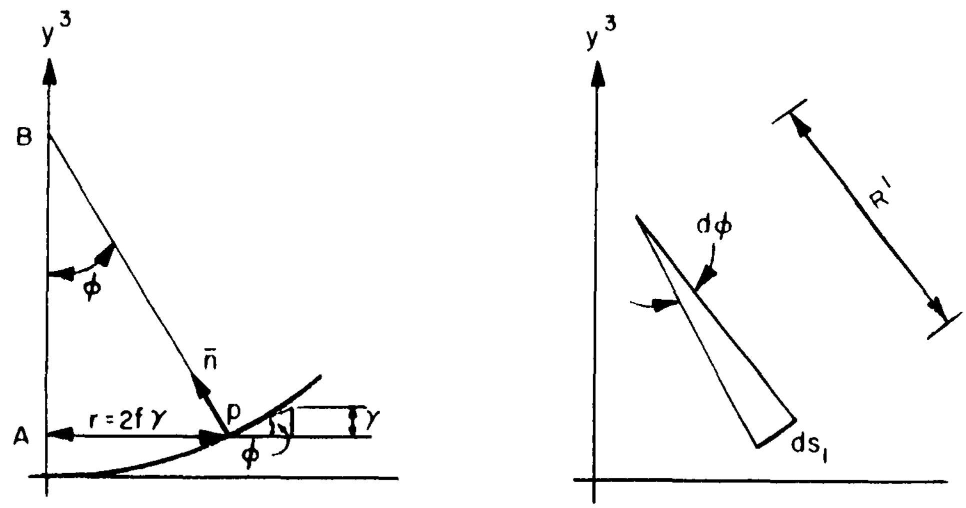


Figure 2.1.4

We observe that the altitude of the triangle ABP is

$$AB = Ap \cot \phi = (2f\gamma)\left(\frac{1}{\gamma}\right) = 2f \qquad 2.1.28$$

and the hypotenuse then can be expressed as

$$BP = \left[(Ap)^2 + (AB)^2\right]^{\frac{1}{2}} = 2f\left[1 + (\gamma)^2\right]^{\frac{1}{2}} \qquad 2.1.29$$

which means that the length $Bp$ is equal to the normal radius of

curvature in the $\theta$ direction, i.e.,

$$BP = R_2 \,. \qquad 2.1.30$$

Additionally, the arc length $ds_1$ , along the $\gamma$ coordinate direction is given by

$$ds_1 = \sqrt{a_{11}}\; d\gamma = 2f\;\left[1+(\gamma)^2\right]^{\frac{1}{2}}\; d\gamma \qquad 2.1.31$$

and therefore, $R_2$ , the radius of curvature is also

$$R_2 = \frac{ds_1}{d\gamma} \,. \qquad 2.1.32$$

We also observe that the arc length, $ds_1$, can be related to $\phi$ and the normal radius of curvature $R_1$ ,

$$ds_1 = R_1\, d\,\phi \qquad 2.1.33$$

or

$$R_1 = \frac{ds_1}{d\phi} \,. \qquad 2.1.34$$

Next we will list the Christoffel symbols of the second kind for the paraboloidal surface. These are given by

$$\left\{ \begin{matrix} \alpha \\ \beta\gamma \end{matrix} \right\}_0 = \bar{a}^{\alpha} \cdot \frac{\partial \bar{a}_\gamma}{\partial \xi^\beta} \qquad 2.1.35$$

where the subscript zero on the Christoffel symbols refers to the undeformed middle surface.

For the present case, the non-zero Christoffel symbols are as follows:

$$\left\{ \begin{matrix} 1 \\ 11 \end{matrix} \right\}_0 = \frac{\gamma}{1+(\gamma)^2} , \qquad 2.1.36$$

$$\left\{ \begin{matrix} 2 \\ 12 \end{matrix} \right\}_0 = \frac{1}{\gamma} , \qquad 2.1.37$$

$$\left\{ \begin{matrix} 1 \\ 22 \end{matrix} \right\}_0 = - \frac{\gamma}{1+(\gamma)^2} . \qquad 2.1.38$$

The following ones are identically zero

$$\left\{ \begin{matrix} 2 \\ 22 \end{matrix} \right\} = 0 , \qquad 2.1.39$$

$$\left\{ \begin{matrix} 1 \\ 12 \end{matrix} \right\} = 0 , \qquad 2.1.40$$

$$\begin{Bmatrix} 2 \\ 11 \end{Bmatrix} = 0. \tag{2.1.41}$$

To complete the picture of the middle surface we require the derivatives of the normal vector and the base vectors. The derivatives of the normal vector are the Weingarten formulae,

$$\frac{\partial \bar{n}}{\partial \xi^{\alpha}} = - a^{\beta\gamma}\, b_{\gamma\alpha}\, \bar{a}_{\beta} \tag{2.1.42}$$

and for the paraboloidal surface these become

$$\frac{\partial \bar{n}}{\partial \theta} = - \frac{1}{R_1}\, \bar{a}_1 = - \frac{\bar{a}_1}{2f\left[1+(\gamma)^2\right]^{3/2}} , \tag{2.1.43}$$

$$\frac{\partial \bar{n}}{\partial \theta} = - \frac{1}{R_2}\, a_2 = - \frac{a_2}{2f\left[1+(\gamma)^2\right]^{1/2}} . \tag{2.1.44}$$

The derivatives of the base vectors are given by the Gauss formulae

$$\frac{\partial \bar{a}_{\alpha}}{\partial \xi^{\beta}} = b_{\alpha\beta}\, \bar{n} + \begin{Bmatrix} \delta \\ \alpha\beta \end{Bmatrix} \bar{a}_{\delta} \tag{2.1.45}$$

and for the paraboloidal surface these become

$$\frac{\partial \bar{a}_1}{\partial \gamma} = \frac{\gamma}{1+(\gamma)^2} \; \bar{a}_1 + \frac{2f}{\left[1+(\gamma)^2\right]^{1/2}} \; \bar{n} \, , \qquad 2.1.46$$

$$\frac{\partial \bar{a}_1}{\partial \theta} = \frac{\partial \bar{a}_2}{\partial \gamma} = \frac{1}{\gamma} \; \bar{a}_2 \, , \qquad 2.1.47$$

$$\frac{\partial \bar{a}_2}{\partial \theta} = -\frac{\gamma}{1+(\gamma)^2} \; \bar{a}_1 + \frac{2f(\gamma)^2}{\left[1+(\gamma)^2\right]^{1/2}} \; \bar{n} \, . \qquad 2.1.48$$

## 2.2 GEOMETRY OF THE MIDDLE SURFACE: CARTESIAN PARAMETERS

In section 2.1, the geometry of the middle surface has been described by means of polar parameters, r and $\theta$. These may even be considered as a "natural" choice since the polar parameters lead to coordinate curves which are also the lines of curvature. There is another choice, which at first glance seems highly "un-natural", which is motivated by the symmetries of the loading experienced by a parabolic antenna. If the predominant loading is attributed to gravity, then it is clear there exists an axis of symmetry and an axis of anti-symmetry.

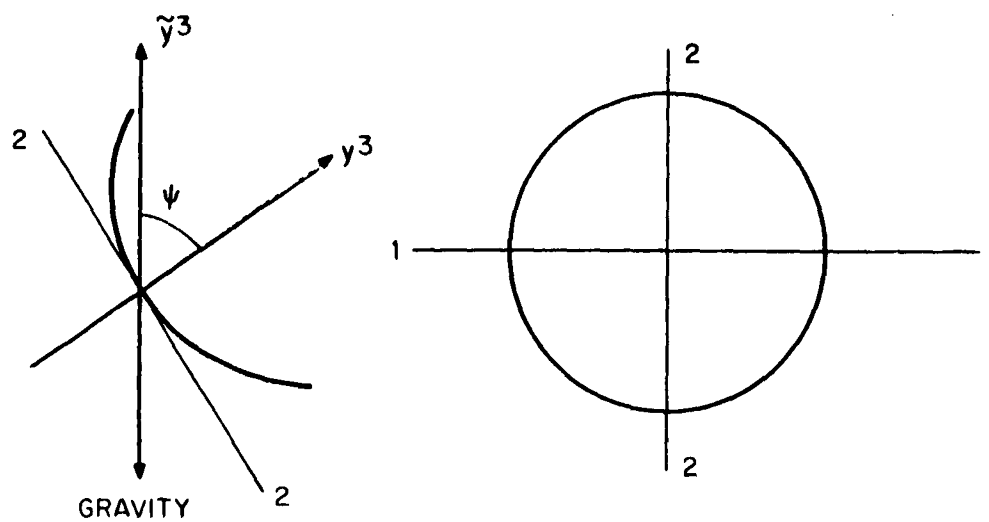


Figure 2.2.1

There is shown in figure 2.2.1 a parabolic reflector with its focal axis inclined at an angle $\psi$ to the gravity axis. The behavior of the antenna shell structure due to the gravity loading is seen to be symmetric with respect to axis 2-2 and anti-symmetric with respect to axis 1-1.

The so-called "natural" coordinates $r$ and $\theta$, which are best suited for problems with rotational symmetry, may not be able to conveniently take advantage of the symmetries of construction which may be motivated by gravity loading. For example, it may be advantageous to reinforce the shell with members which are parallel to the 2-2 axis. Accordingly, it is felt that the use of cartesian parameters may be fruitful. These are defined as follows (cf. equation 2.1.1 and figure 2.2.2):

$$y^1 = 2fx \quad , \qquad 2.2.1$$

$$y^2 = 2fy \quad , \qquad 2.2.2$$

$$y^3 = f\left[(x)^2 + (y)^2\right] \quad . \qquad 2.2.3$$

It should be observed that x and y are non-dimensional parameters on the surface.

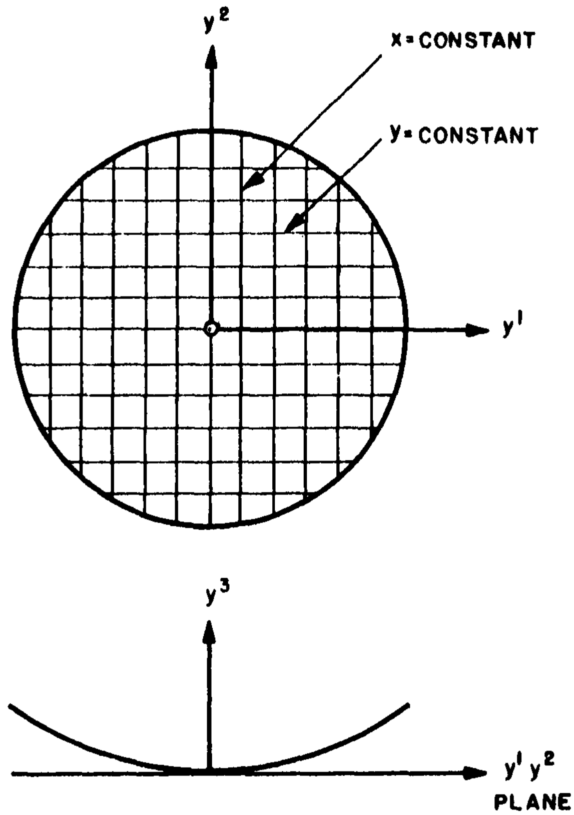


Figure 2.2.2

The position vector to points on the middle surface is now written as

$$\bar{r}_0(x,y) = 2fx\,\bar{i}_1 + 2fy\,\bar{i}_2 + f\left[(x)^2+(y)^2\right]\bar{i}_3 \; . \qquad 2.2.4$$

We summarize without comment the base vectors, first fundamental tensor, second fundamental tensor, and Christoffel symbols of the second kind.

$$\bar{a}_1 = 2f\bar{i}_1 + 2fx\bar{i}_3 \quad , \qquad 2.2.5$$

$$\bar{a}_2 = 2f\bar{i}_2 + 2fy\bar{i}_3 \quad , \qquad 2.2.6$$

$$a_{11} = 4f^2(1+x^2) \quad , \qquad 2.2.7$$

$$a_{12} = 4f^2 xy \quad , \qquad 2.2.8$$

$$a_{22} = 4f^2(1+y^2) \quad , \qquad 2.2.9$$

$$a = 16f^4(1+x^2+y^2) \quad , \qquad 2.2.10$$

$$\bar{n} = \frac{1}{\sqrt{a}}\left[-4f^2x\bar{i}_1 - 4f^2y\bar{i}_2 + 4f^2\bar{i}_3\right] , \qquad 2.2.11$$

$$b_{11} = \frac{2f}{\sqrt{1+x^2+y^2}} \quad , \qquad 2.2.12$$

$$b_{12} = 0 \quad , \qquad 2.2.13$$

$$b_{22} = \frac{2f}{\sqrt{1+x^2+y^2}} \quad , \qquad 2.2.14$$

$$a^{11} = \frac{1+y^2}{4f^2(1+x^2+y^2)} \quad , \qquad 2.2.15$$

$$a^{12} = -\frac{xy}{4f^2(1+x^2+y^2)} \quad , \qquad 2.2.16$$

$$a^{22} = \frac{1+x^2}{4f^2(1+x^2+y^2)} \quad , \qquad 2.2.17$$

$$\left\{ {1 \atop 11} \right\}_0 = \frac{16f^4}{a} x = \frac{x}{1+x^2+y^2} \quad , \qquad 2.2.18$$

$$\left\{ {1 \atop 22} \right\}_0 = \frac{16f^4}{a} x = \frac{x}{1+x^2+y^2} \quad , \qquad 2.2.19$$

$$\left\{ {2 \atop 22} \right\}_0 = \frac{16f^4}{a} y = \frac{y}{1+x^2+y^2} \quad , \qquad 2.2.20$$

$$\left\{ {2 \atop 11} \right\}_0 = \frac{16f^4}{a} y = \frac{y}{1+x^2+y^2} \quad , \qquad 2.2.21$$

$$\left\{ {1 \atop 21} \right\}_0 = \left\{ {1 \atop 12} \right\}_0 = \left\{ {2 \atop 12} \right\}_0 = \left\{ {2 \atop 21} \right\}_0 = 0 \; , \qquad 2.2.22$$

$$b_1^1 = \frac{1+(y)^2}{2f[1+x^2+y^2]^{3/2}} , \qquad 2.2.23$$

$$b_2^1 = \frac{xy}{2f[1+x^2+y^2]^{3/2}} , \qquad 2.2.24$$

$$b_1^2 = \frac{xy}{2f[1+x^2+y^2]^{3/2}} , \qquad 2.2.25$$

$$b_2^2 = \frac{1+(x)^2}{2f[1+x^2+y^2]^{3/2}} , \qquad 2.2.26$$

$$\frac{\partial \bar{a}_1}{\partial x} = \frac{2f}{\sqrt{1+x^2+y^2}} \bar{n} + \frac{x}{1+x^2+y^2} \bar{a}_1 + \frac{y}{1+x^2+y^2} \bar{a}_2 , \qquad 2.2.27$$

$$\frac{\partial \bar{a}_2}{\partial x} = \frac{\partial \bar{a}_1}{\partial y} = 0 , \qquad 2.2.28$$

$$\frac{\partial \bar{a}_2}{\partial y} = \frac{2f}{\sqrt{1+x^2+y^2}} \bar{n} + \frac{x}{1+x^2+y^2} \bar{a}_1 + \frac{y}{1+x^2+y^2} \bar{a}_2 , \qquad 2.2.29$$

$$\frac{\partial \bar{n}}{\partial x} = - \frac{1+y^2}{2f[1+x^2+y^2]^{\frac{3}{2}}} \bar{a}_1 - \frac{xy}{2f[1+x^2+y^2]^{\frac{3}{2}}} \bar{a}_2 , \qquad 2.2.30$$

$$\frac{\partial \bar{n}}{\partial y} = - \frac{xy}{2f[1+x^2+y^2]^{3/2}} \bar{a}_1 - \frac{1+x^2}{2f[1+x^2+y^2]^{3/2}} \bar{a}_2 \, . \qquad 2.2.31$$

It should be observed that the x, y parameters are non-orthogonal ( $a_{12} \neq 0$) but are conjugate ($b_{12} = 0$).

There is one other geometrical property of the paraboloid which is of interest. The curve formed by the intersection of the paraboloid and the plane $y^1 = \ell$ is the same as the generating parabola (see figure 2.2.3). This can be demonstrated

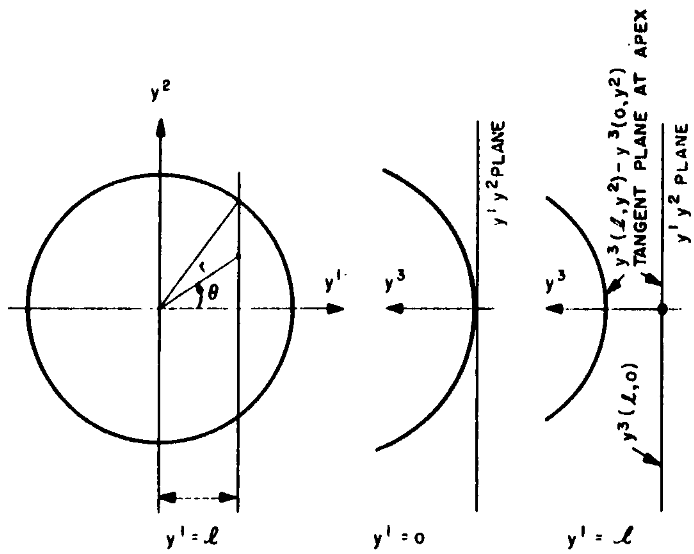


Figure 2.2.3

as follows:

The generating parabola (see equation 2.1.1) is

$$y^3 = \frac{(r)^2}{4f} \quad . \tag{2.2.32}$$

At $y^1 = 0$ ,

$$r = y^2 \tag{2.2.33}$$

and hence the intersection of the paraboloid and the plane $y^1 = 0$ has the representation

$$y^3(0, y^2) = \frac{(y^2)^2}{4f} \quad . \tag{2.2.34}$$

The slope of the curve is

$$\frac{dy^3(0, y^2)}{dy^2} = \frac{y^2}{4f} \quad . \tag{2.2.35}$$

At $y^1 = \ell$ ,

$$r = \sqrt{(\ell)^2 + (y^2)^2} \tag{2.2.36}$$

and hence the intersection of the paraboloid and the plane $y^1 = \ell$ is represented by

$$y^3(\ell, y^2) = \frac{(\ell)^2 + (y^2)^2}{4f} \, . \qquad 2.2.37$$

The apex of this curve is displaced from the $y^1 y^2$ plane by an amount

$$y^3(\ell, 0) = \frac{(\ell)^2}{4f} \qquad 2.2.38$$

and hence if we use the tangent plane at the apex as a reference, the curve is represented by the formula

$$y^3(\ell, y^2) - y^3(\ell, 0) = \frac{(y^2)^2}{4f} \, . \qquad 2.2.39$$

This is the same as formula 2.2.34. Furthermore the slope of the curve

$$\frac{dy^3(\ell, y^2)}{dy^2} = \frac{y^3}{2f} \qquad 2.2.40$$

is the same as that given by equation 2.2.35. Thus, we have shown that the curve of the surface along $y^1 = \ell$ is precisely the generating parabola displaced by an amount $y^3(\ell, 0)$.

## 2.3 GEOMETRY OF THE SHELL: POLAR PARAMETERS

The location of a point within the shell structure will be specified by three parameters, two of which lie on the middle surface while the third is oriented along the normal to the middle surface. These parameters are denoted by $\gamma$ , $\theta$ ,and $\zeta$ respectively and are shown in figure 2.3.1.

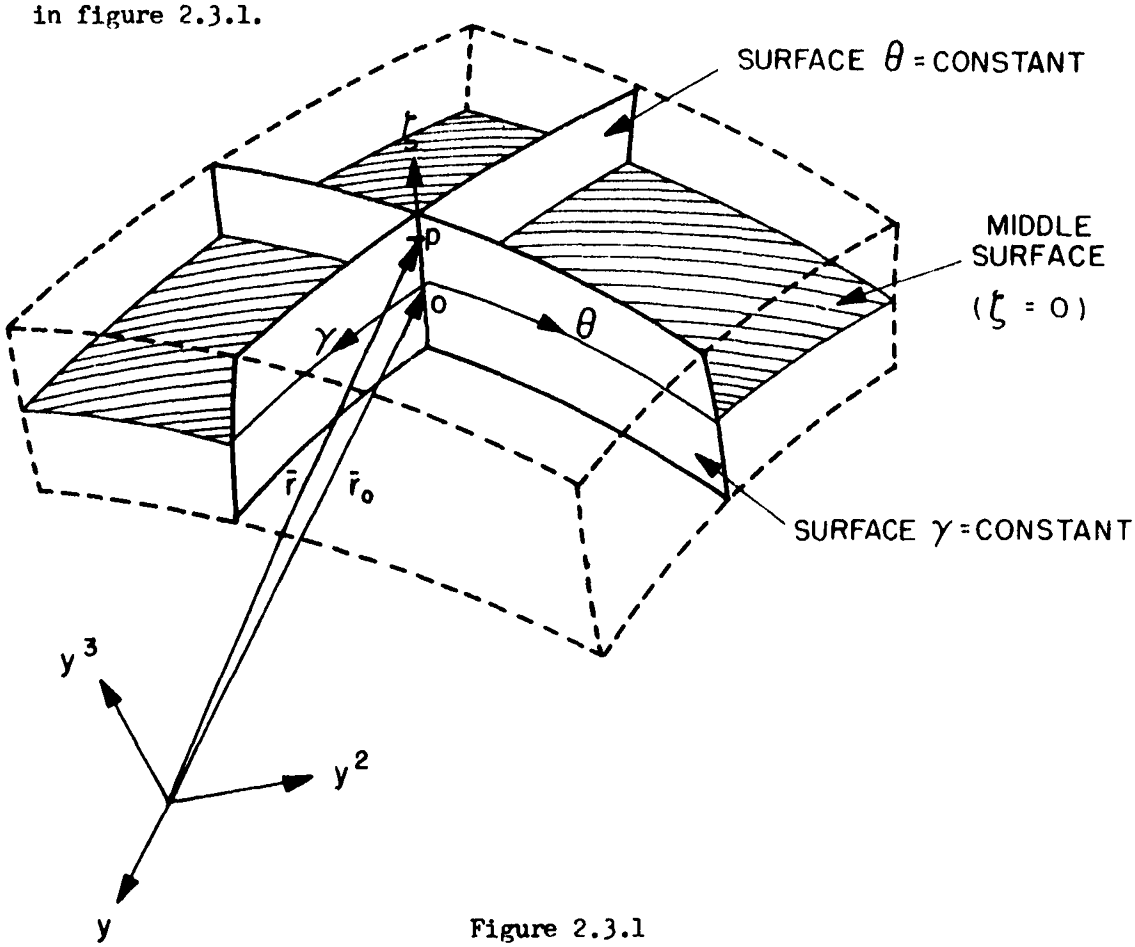


Figure 2.3.1

We see from figure 2.3.1 that point o is on the middle surface and that p which is not on the middle surface has the position vector

$$\bar{r}(\gamma,\theta,\zeta) = \bar{r}_0(\gamma,\theta) + \zeta\bar{n} \quad . \qquad 2.3.1$$

At point p, the base vectors $\bar{g}_m$ are

$$\bar{g}_1 = \frac{\partial \bar{r}}{\partial \gamma} = \left(1 - \frac{\zeta}{R_1}\right)\bar{a}_1 \quad , \qquad 2.3.2$$

$$\bar{g}_2 = \frac{\partial \bar{r}}{\partial \theta} = \left(1 - \frac{\zeta}{R_2}\right)\bar{a}_2 \quad , \qquad 2.3.3$$

$$\bar{g}_3 = \frac{\partial \bar{r}}{\partial \zeta} = \bar{n} \qquad 2.3.4$$

where $R_1$, $R_2$ are the principal radii of curvature associated with the middle surface (see equations 2.1.26 and 2.1.27) of the paraboloid, and $\bar{a}_1$ , $\bar{a}_2$ are the base vectors of the middle surface, (see equations 2.1.8 and ?.1.9).

The non-zero components of the covariant fundamental tensor are

$$g_{11} = \left(1-\frac{\zeta}{R_1}\right)^2 a_{11} = \left\{1-\frac{\zeta}{2f[1+(\gamma)^2]^{3/2}}\right\}^2 4(f)^2[1+(\gamma)^2] \,, \tag{2.3.5}$$

$$g_{22} = \left(1-\frac{\zeta}{R_2}\right)^2 a_{22} = \left\{1-\frac{\zeta}{2f[1+(\gamma)^2]^{1/2}}\right\}^2 4(f)^2(\gamma)^2 \,, \tag{2.3.6}$$

$$g_{33} = 1 \tag{2.3.7}$$

and the determinant g has the value

$$g = \left(1-\frac{\zeta}{R_1}\right)^2\left(1-\frac{\zeta}{R_2}\right)^2 a_{11}\, a_{22}$$

$$= \left\{1-\frac{\zeta}{2f[1+(\gamma)^2]^{3/2}}\right\}^2\left\{1-\frac{\zeta}{2f[1+(\gamma)^2]^{1/2}}\right\}^2 16(f)^4[1+(\gamma)^2](\gamma)^2 . \tag{2.3.8}$$

We also list for future use the associated contravariant quantities in terms of the middle surface parameters.

$$\bar{g}^1 = \frac{\bar{a}_1}{a_{11}\left(1-\frac{\zeta}{R_1}\right)} = \frac{\bar{a}^1}{\left(1-\frac{\zeta}{R_1}\right)} \,, \tag{2.3.9}$$

$$\bar{g}^2 = \frac{\bar{a}_2}{a_{22}\left(1-\frac{\zeta}{R_2}\right)} = \frac{\bar{a}^2}{\left(1-\frac{\zeta}{R_2}\right)} \,, \tag{2.3.10}$$

$$\bar{g}^3 = \bar{n} \ , \qquad 2.3.11$$

$$g^{11} = \frac{1}{a_{11}\left(1-\frac{\zeta}{R_1}\right)^2} \ , \qquad 2.3.12$$

$$g^{22} = \frac{1}{a_{22}\left(1-\frac{\zeta}{R_2}\right)^2} \ , \qquad 2.3.13$$

$$g^{33} = 1 \ . \qquad 2.3.14$$

To complete the picture, the Christoffel symbols of the second kind are tabulated for the space occupied by the shell structure.

$$\left\{ {1 \atop 11} \right\} = \frac{\gamma}{1+(\gamma)^2} \ , \qquad 2.3.15$$

$$\left\{ {2 \atop 12} \right\} = \frac{1}{\gamma} \ , \qquad 2.3.16$$

$$\left\{ {1 \atop 22} \right\} = -\frac{\left(1-\frac{\zeta}{R_2}\right)^2}{\left(1-\frac{\zeta}{R_1}\right)^2} \ \frac{\gamma}{1+(\gamma)^2} \ , \qquad 2.3.17$$

$$\left\{ {1 \atop 13} \right\} = \left\{ {1 \atop 31} \right\} = - \frac{1}{\left(1 - \frac{\zeta}{R_1}\right) R_1} \tag{2.3.18}$$

$$\left\{ {2 \atop 23} \right\} = - \frac{1}{\left(1 - \frac{\zeta}{R_2}\right) R_2} \tag{2.3.19}$$

$$\left\{ {3 \atop 11} \right\} = \frac{4(f)^2 \left[1 + (\gamma)^2\right]}{R_1} \left(1 - \frac{\zeta}{R_1}\right) \tag{2.3.20}$$

$$\left\{ {3 \atop 22} \right\} = \frac{4(f)^2 (\gamma)^2}{R_2} \left(1 - \frac{\zeta}{R_2}\right) \tag{2.3.21}$$

$$\left\{ {2 \atop 22} \right\} = \left\{ {1 \atop 12} \right\} = \left\{ {2 \atop 11} \right\} = \left\{ {1 \atop 23} \right\} = \left\{ {\alpha \atop 33} \right\} = \left\{ {3 \atop \alpha 3} \right\} = 0 \tag{2.3.22}$$

## 2.4 GEOMETRY OF THE SHELL: CARTESIAN PARAMETERS

We will proceed as in section 2.3 which describes the geometry of the shell in terms of polar parameters. The position vector to a point with coordinates x, y, $\zeta$ is represented by

$$\bar{r}(x, y, \zeta) = \bar{r}_0 (x, y) + \zeta \bar{n} . \qquad 2.4.1$$

At point p, the base vectors are

$$\bar{g}_1 = \frac{\partial \bar{r}}{\partial x} = \left(1 - \frac{1+y^2}{2f[1+x^2+y^2]^{3/2}}\right) \bar{a}_1 - \left(\frac{xy}{2f[1+x^2+y^2]^{3/2}}\right) \bar{a}_2, \qquad 2.4.2$$

$$\bar{g}_2 = \frac{\partial \bar{r}}{\partial y} = \left(1 - \frac{1+x^2}{2f(1+x^2+y^2)^{3/2}}\right) \bar{a}_2 - \left(\frac{xy}{2f(1+x^2+y^2)^{3/2}}\right) \bar{a}_1, \qquad 2.4.3$$

$$\bar{g}_3 = \frac{\partial \bar{r}}{\partial \zeta} = n. \qquad 2.4.4$$

The remaining geometrical properties can be calculated but will not be tabulated since it is evident the non-orthogonal nature of the x, y coordinates will lead to rather unwieldly expressions.

## III ANALYSIS OF DEFORMATION

### 3.1 THE STRAIN TENSOR

The square of the line-element, ds, in the undeformed structure of the shell will be written as

$$\left(ds\right)^2 = g_{mn}\, dx^m\, dx^n \tag{3.1.1}$$

where $g_{mn}$ is the fundamental metric covariant tensor of the space occupied by the undeformed structure and $x^1$, $x^2$, $x^3$ are the curvilinear coordinates which locate a specific point in the undeformed structure. We will also use $x^1$, $x^2$, $x^3$, as parameters to locate points in the deformed structure and hence the square of the line-element, dS, in the deformed structure will be written as

$$\left(dS\right)^2 = G_{mn}\, dx^m\, d^n \tag{3.1.2}$$

where $G_{mn}$ is the fundamental metric covariant tensor of the deformed structure.

The state of strain is characterized by the difference of the square of the line-elements and will be written as

$$(dS)^2-(ds)^2=2\,\gamma_{mn}\,dx^m\,dx^n \qquad 3.1.3$$

where it can be observed from an examination of equations 3.1.1 and 3.1.2 that

$$\gamma_{mn}=\frac{1}{2}\left(G_{mn}-g_{mn}\right) \qquad 3.1.4$$

The second order covariant tensor, $\gamma_{mn}$, defined by equation 3.1.3 is the strain tensor. Since equation 3.1.3 is written in terms of the Lagrangian coordinates, i.e., the coordinates of the undeformed body, the strain tensor, $\gamma_{mn}$, is sometimes referred to as the Lagrangian strain tensor.

The metric tensor, $G_{mn}$, of the deformed structure is related to the metric tensor of the undeformed structure displacement vector, $\bar{V}$ .

Let $\bar{r}$ be the position vector to a point, p, in the undeformed body and $\bar{R}$ be the position vector to P, the point which p occupies after deformation. Then

$$\bar{R} = \bar{r} + \bar{v} \qquad 3.1.5$$

and the base vectors of the deformed structure are given by

$$\bar{G}_m = \frac{\partial \bar{R}}{\partial x^m} = \frac{\partial \bar{r}}{\partial x^m} + \frac{\partial \bar{v}}{\partial x^m} = \bar{g}_m + \frac{\partial \bar{v}}{\partial x^m} \quad . \qquad 3.1.6$$

It is easily verified that the components of the strain tensor can be calculated from the vector equation

$$\gamma_{mn} = \frac{1}{2}\left(\bar{g}_m \cdot \frac{\partial \bar{v}}{\partial x^m} + \bar{g}_n \cdot \frac{\partial \bar{v}}{\partial x^m} + \frac{\partial \bar{v}}{\partial x^m} \cdot \frac{\partial \bar{v}}{\partial x^n}\right) \cdot \qquad 3.1.7$$

## 3.2 THE COMPONENTS OF LARGE STRAIN

In this section we will specialize the results of the preceding section to a coordinate system which is more suited to the theory of shells than are the general curvilinear coordinates $x^m$. We will designate these to be $\xi^1$, $\xi^2$, and $\zeta$ at this stage of the analysis. In the later sections $\xi^1$ and $\xi^2$ will take on the roles of $r$ and $\theta$ in the case of the polar parameters (see section 2.4).

The displacement vector, $\bar{v}$, will be specified in terms of components aligned with respect to the base vectors, $\bar{a}_\alpha$, of the undeformed middle surface and the normal, $\bar{n}$, to the undeformed middle surface. We express this as

$$\bar{v}\left(\xi^1,\xi^2,\zeta\right) = v^\sigma\left(\xi^1,\xi^2,\zeta\right)\bar{a}_\sigma + v^3\left(\xi^1,\xi^2,\zeta\right)\bar{n}\,. \qquad 3.2.1$$

The six components of the covariant strain tensor in the shell coordinates (see equation 3.1.7) are determined by the following vector equations:

$$\gamma_{\alpha\beta} = \frac{1}{2}\left(\bar{g}_\alpha \cdot \frac{\partial \bar{v}}{\partial \xi^\beta} + \bar{g}_\beta \cdot \frac{\partial \bar{v}}{\partial \xi^\alpha} + \frac{\partial \bar{v}}{\partial \xi^\alpha} \cdot \frac{\partial \bar{v}}{\partial \xi^\beta}\right), \qquad 3.2.2$$

$$\gamma_{\alpha 3} = \frac{1}{2}\left(\bar{g}_\alpha \cdot \frac{\partial \bar{v}}{\partial \zeta} + \bar{n} \cdot \frac{\partial \bar{v}}{\partial \xi^\alpha} + \frac{\partial \bar{v}}{\partial \zeta} \cdot \frac{\partial \bar{v}}{\partial \xi^\alpha}\right), \qquad 3.2.3$$

$$\gamma_{33} = \bar{n} \cdot \frac{\partial \bar{v}}{\partial \zeta} + \frac{1}{2}\,\frac{\partial \bar{v}}{\partial \zeta} \cdot \frac{\partial \bar{v}}{\partial \zeta}\,. \qquad 3.2.4$$

Let us write out in explicit form the six components of the strain tensor if the lines of curvature are utilized as the coordinates on the middle surface of the undeformed shell.

$$\begin{aligned}\gamma_{11} = {} & \left(v^1_{,1} - \frac{v^3}{R_1}\right)\left(1 - \frac{\zeta}{R_1}\right) a_{11} \\ & + \frac{1}{2}\left\{\left(v^1_{,1} - \frac{v^3}{R_1}\right)^2 a_{11} + \left(v^2_{,1}\, v^2_{,1}\right) a_{22} + \left(\frac{a_{11} v^1}{R_1} + \frac{\partial v^3}{\partial \xi^1}\right)^2\right\},\end{aligned} \qquad 3.2.5$$

$$\begin{aligned}\gamma_{22} = {} & \left(v^2_{,2} - \frac{v^3}{R_2}\right)\left(1 - \frac{\zeta}{R_2}\right) a_{22} \\ & + \frac{1}{2}\left\{\left(v^1_{,1}\, v^1_{,2}\right) a_{11} + \left(v^2_{,2} - \frac{v^3}{R_2}\right)^2 a_{22} + \left(\frac{a_{22} v^2}{R_2} + \frac{\partial v^3}{\partial \xi^2}\right)^2\right\},\end{aligned} \qquad 3.2.6$$

$$\gamma_{12} = \frac{1}{2}\left\{ \left(v^1_{,2}\right)\left(1-\frac{\zeta}{R_1}\right)a_{11} + \left(v^2_{,1}\right)\left(1-\frac{\zeta}{R_2}\right)a_{22} \right.$$

$$+ \left(v^2_{,2} - \frac{v^3}{R_2}\right)\left(v^2_{,1}\right)a_{22} + \left(v^1_{,2}\right)\left(v^1_{,1} - \frac{v^3}{R_1}\right)a_{11}$$

$$\left. + \left(\frac{a_{22}v^2}{R_2} + \frac{\partial v^3}{\partial \xi^2}\right)\left(\frac{a_{11}v^1}{R_1} + \frac{\partial v^3}{\partial \xi^1}\right)\right\}, \qquad 3.2.7$$

$$\gamma_{13} = \frac{1}{2}\left\{\left(\frac{\partial v^1}{\partial \zeta}\right)\left(1-\frac{\zeta}{R_1}\right)a_{11} + \frac{a_{11}v^1}{R_1} + \frac{\partial v^3}{\partial \xi^1}\right.$$

$$+ \left(\frac{\partial v^1}{\partial \zeta}\right)\left(v^1_{,1} - \frac{v^3}{R_1}\right)a_{11} + \left(\frac{\partial v^2}{\partial \zeta}\right)\left(v^2_{,1}\right)a_{22}$$

$$\left. + \left(\frac{\partial v^3}{\partial \zeta}\right)\left(\frac{a_{11}v^1}{R_1} + \frac{\partial v^3}{\partial \xi^1}\right)\right\}, \qquad 3.2.8$$

$$\gamma_{23} = \frac{1}{2}\left\{\left(\frac{\partial v^2}{\partial \zeta}\right)\left(1-\frac{\zeta}{R_2}\right)a_{22} + \frac{a_{22}v^2}{R_2} + \frac{\partial v^3}{\partial \xi^2}\right.$$

$$+ \left(\frac{\partial v^2}{\partial \zeta}\right)\left(v^2_{,2} - \frac{v^3}{R_2}\right)a_{22} + \left(\frac{\partial v^1}{\partial \zeta}\right)\left(v^1_{,2}\right)a_{11}$$

$$\left. + \left(\frac{\partial v^3}{\partial \zeta}\right)\left(\frac{a_{22}v^2}{R_2} + \frac{\partial v^3}{\partial \xi^2}\right)\right\}, \qquad 3.2.9$$

$$\gamma_{33} = \frac{\partial v^3}{\partial \zeta} + \frac{1}{2}\left\{\frac{\partial v^1}{\partial \zeta}\frac{\partial v^1}{\partial \zeta}a_{11} + \frac{\partial v^2}{\partial \zeta}\frac{\partial v^2}{\partial \zeta}a_{22} + \left(\frac{\partial v^3}{\partial \zeta}\right)^2\right\}. \qquad 3.2.10$$

It should be observed that none of the usual restrictions of smallness, thinness and linearity have as yet been applied.

## 3.3 THE COMPONENTS OF INFINITESIMAL STRAIN FOR THIN SHELLS

The thinness of the shell is specified by the ratio of its thickness to the smallest radius of curvature of its middle surface. If we let h be the thickness of the shell, then "thinness" can be expressed as

$$\frac{\zeta}{R_1}, \frac{\zeta}{R_2} << 1 \qquad -\frac{h}{2} \leq \zeta \leq \frac{h}{2} \tag{3.3.1}$$

Hence, the term $\left(1 - \frac{\zeta}{R_{i\alpha}}\right)$, which appears in each of the expressions derived in the previous section, will be replaced by 1 wherever it appears.

The infinitesimal nature of the strains means that the elongations

$$E_n = \sqrt{1 + \frac{2\gamma_{nn}}{g_{nn}}} - 1 \quad (\text{no sum on } n) \tag{3.3.2}$$

and the changes in angles between directions $\bar{g}_m$ and $\bar{g}_n$

$$\sin \phi_{mn} = \frac{2\gamma_{mn}}{\sqrt{g_{mm}\, g_{nn}\, (1+E_m)(1+E_n)}} \tag{3.3.3}$$

are not only small but are infinitesimal. If the strains are small, then

$$E_n \approx \frac{\gamma_{nn}}{g_{nn}}, \qquad 3.3.4$$

$$\sin \phi_{mn} \approx \phi_{mn} \approx \frac{2\gamma_{mn}}{\sqrt{g_{mm}\, g_{nn}}}. \qquad 3.3.5$$

The added restriction to infinitesimal strain enables us to neglect the non-linear terms in the strain-displacement relations. In addition to these simplifications, let us introduce the assumption that points lying along the normal to the undeformed shell remain on a straight line in the deformed shell. This is expressed as

$$V^{\sigma}(\xi^1, \xi^2, \zeta) = u^{\sigma}(\xi^1, \xi^2) + \zeta\, w^{\sigma}(\xi^1, \xi^2). \qquad 3.3.6$$

A companion assumption is one which states that $V^3$, the displacement normal to the surface, is a function only of $\xi^1$ and $\xi^2$, i.e.,

$$V^3(\xi^1, \xi^2, \zeta) = w(\xi^1, \xi^2). \qquad 3.3.7$$

We will use $\epsilon_{nm}$ to denote the infinitesimal strain tensor, i.e.,

$$\epsilon_{nm} \approx \gamma_{nm} \qquad 3.3.8$$

The strain-displacement relations for the case of a thin shell and infinitesimal strain are as follows:

$$\gamma_{11} = (u^1_{,1} - b^1_1 w)\, a_{11} + (u^2_{,1} - b^2_1 w)\, a_{12} + \zeta(\omega^1_{,1}\, a_{11} + \omega^2_{,1}\, a_{22}), \qquad 3.3.9$$

$$\gamma_{22} = (u^1_{,2} - b^1_2 w)\, a_{12} + (u^2_{,2} - b^2_2\, w)\, a_{22} + \zeta(\omega^1_{,2}\, a_{12} + \omega^2_{,2}\, a_{22}), \qquad 3.3.10$$

$$\begin{aligned}\gamma_{12} = \frac{1}{2}\Big\{ &(u^1_{,2} - b^1_2\, w)\, a_{11} + (u^2_{,2} - b^2_2\, w)\, a_{12} \\ &+ (u^1_{,1} - b^1_1\, w)\, a_{12} + (u^2_{,1} - b^2_1\, w)\, a_{22} \\ &+ \zeta(\omega^1_{,2}\, a_{11} + \omega^2_{,2}\, a_{12} + \omega^1_{,1}\, a_{12} + \omega^2_{,1}\, a_{22})\Big\},\end{aligned} \qquad 3.3.11$$

$$\gamma_{23} = \frac{1}{2}\left\{\omega^1 a_{12} + \omega^2 a_{22} + u^1 b_{12} + u^2 b_{22} + \frac{\partial w}{\partial \xi^2} + \zeta(\omega^1 b_{12} + \omega^3 b_{22})\right\}, \qquad 3.3.12$$

$$\gamma_{31} = \frac{1}{2}\left\{\omega^1 a_{11} + \omega^2 a_{12} + u^1 b_{11} + u^2 b_{21} + \frac{\partial w}{\partial \xi^1} + \zeta(\omega^1 b_{11} + \omega^2 b_{21})\right\}. \qquad 3.3.13$$

There are five generalized displacements in equations 3.3.9 to 3.3.13. These are $u^1$, $u^2$, $w$, $\omega^1$, and $\omega^2$. The first three are directly proportional to displacements of the middle surface whereas the last two are proportional to the angle of rotation which the normal to the deformed surface undergoes during deformation. These latter two ($\omega^1$ and $\omega^2$) are a first approximation to the effect of transverse shearing deformations.

In a large majority of problems, the effects of transverse shearing deformations are negligible and we can place a further restriction on the distortions assuming that

$$\gamma_{13} \cong 0, \qquad 3.3.14$$

$$\gamma_{23} \cong 0. \qquad 3.3.15$$

This permits us to eliminate $\omega_1$ and $\omega_2$ since equations 3.3.12 and 3.3.13 can now be solved to yield

$$\omega_1 = -b_{11}\, u^1 - b_{21}\, u^2 - \frac{\partial w}{\partial \xi^1}, \qquad 3.3.16$$

$$\omega^2 = -b_{12}u^1 - b_{22}\,u^2 - \frac{\partial w}{\partial \xi^2}\,. \qquad 3.3.17$$

It is observed that the strain-displacement relations consist of a term which represents the stretching of the middle surface and an additional term due to bending of the middle surface. Hence we write

$$\gamma_{\alpha\beta}\,(\xi^1, \xi^2, \zeta) = \gamma^{o}_{\alpha\beta}\,(\xi^1, \xi^2) + \zeta k_{\alpha\beta}\,(\xi^1, \xi^2) \qquad 3.3.18$$

where it can be shown that

$$\gamma^{o}_{\alpha\beta} = \frac{1}{2}\left(u_{\alpha,\beta} + u_{\beta,\alpha} - 2b_{\alpha\beta}\,w\right), \qquad 3.3.19$$

$$k_{\alpha\beta} = \frac{1}{2}\left(\omega_{\alpha,\beta} + \omega_{\beta,\alpha}\right). \qquad 3.3.20$$

We will refer to $\gamma^{o}_{\alpha\beta}$ as the strain tensor of the middle surface and to $k_{\alpha\beta}$ as the strain-curvature tensor of the middle surface.

## 3.4 STRAIN-DISPLACEMENT RELATIONS: POLAR PARAMETERS

In this section we will apply the formulae of section 3.3 to the paraboloid as described by the polar parameters, r and $\theta$, which are also the lines of curvature. The results in terms of the tensor components are as follows:

$$\gamma_{11}^{0} = \frac{\partial u_1}{\partial \gamma} - \frac{\gamma}{1+(\gamma)^2} u_1 - \frac{2f}{\sqrt{1+(\gamma)^2}} w, \qquad 3.4.1$$

$$\gamma_{22}^{0} = \frac{\partial u_2}{\partial \theta} + \frac{\gamma}{1+(\gamma)^2} u_1 - \frac{2f(\gamma)^2}{\sqrt{1+(\gamma)^2}}, \qquad 3.4.2$$

$$\gamma_{12}^{0} = \frac{1}{2}\left\{ \frac{\partial u_1}{\partial \theta} + \frac{\partial u_2}{\partial \gamma} - \frac{2}{\gamma} u_2 \right\} \qquad 3.4.3$$

$$k_{11} = \frac{\partial \omega_1}{\partial \gamma} - \frac{\gamma}{1+(\gamma)^2} \omega_1, \qquad 3.4.4$$

$$k_{22} = \frac{\partial \omega_2}{\partial \theta} + \frac{\gamma}{1+(\gamma)^2} \omega_1, \qquad 3.4.5$$

$$k_{12} = \frac{1}{2}\left\{\frac{\partial \omega_1}{\partial \theta} + \frac{\partial \omega_2}{\partial \gamma} - \frac{2}{\gamma}\ \omega_2\right\}, \qquad 3.4.6$$

$$\omega_1 = -\frac{2f}{\sqrt{1+(\gamma)^2}}\ u^1 - \frac{\partial w}{\partial \gamma}, \qquad 3.4.7$$

$$\omega_2 = -\frac{2f(\gamma)^2}{\sqrt{1+(\gamma)^2}}\ u^2 - \frac{\partial w}{\partial \theta}. \qquad 3.4.8$$

Let us introduce at this stage the physical components of the strain tensors into the strain-displacement relations. Also, we will switch to the $r, \theta$ notation for the components of strain and displacements. We note that

$$\epsilon_r \equiv \epsilon_{11} = \frac{\gamma_{11}}{a_{11}}, \qquad 3.4.9$$

$$\epsilon_\theta \equiv \epsilon_{22} = \frac{\gamma_{22}}{a_{22}}, \qquad 3.4.10$$

$$\epsilon_{r\theta} \equiv \epsilon_{12} = \frac{\gamma_{12}}{\sqrt{a_{11}a_{22}}}. \qquad 3.4.11$$

We will also at this stage separate the strain into two portions, one of which is the strain of the middle surface and the other which is proportional to $\zeta$ . To this purpose we write

$$\epsilon_r = \epsilon_r^o + \zeta K_r , \tag{3.4.12}$$

$$\epsilon_\theta = \epsilon_\theta^o + \zeta K_\theta , \tag{3.4.13}$$

$$\epsilon_{r\theta} = \epsilon_{r\theta}^o + \zeta K_{r\theta} . \tag{3.4.14}$$

Before we calculate $\epsilon_{\alpha\beta}^o$ and $K_{\alpha\beta}^o$ , let us introduce the physical components of $u^\alpha$ and $\omega^\alpha$ . These are given for the case of the orthogonal coordinates by

$$u_r^o = \sqrt{a_{11}}\; u^1 = \sqrt{a^{11}}\; u_1 , \tag{3.4.15}$$

$$u_\theta^o = \sqrt{a_{22}}\; u^2 = \sqrt{a^{22}}\; u_2 , \tag{3.4.16}$$

$$\omega_r^o = \sqrt{a_{11}}\ \omega^1 = \sqrt{a^{11}}\ \omega_1 , \qquad 3.4.17$$

$$\omega_\theta^o = \sqrt{a_{22}}\ \omega^2 = \sqrt{a^{22}}\ \omega_2 \qquad 3.4.18$$

where $u_r^o$, $u_\theta^o$, $\omega_r^o$ and $\omega_\theta^o$ denote the physical components of displacement and rotation.

The pertinent strain-displacement relations for the paraboloidal shell described by polar parameters are as follows:

$$\epsilon_r^o = \frac{1}{2f\sqrt{1+\gamma^2}} \frac{\partial u_r^o}{\partial r} - \frac{w}{2f\left[1+\gamma^2\right]^{3/2}} , \qquad 3.4.19$$

$$\epsilon_\theta^o = \frac{1}{2f\gamma} \frac{\partial u_\theta^o}{\partial \theta} + \frac{u_r^o}{2f\gamma\sqrt{1+\gamma^2}} - \frac{w}{2f\sqrt{1+\gamma^2}} \qquad 3.4.20$$

$$\epsilon_{r\theta}^o = \frac{1}{2f\sqrt{1+\gamma^2}} \frac{\partial u_\theta^o}{\partial r} - \frac{u_\theta^o}{2f\gamma\sqrt{1+\gamma^2}} + \frac{1}{2f\gamma} \frac{\partial u_r^o}{\partial \theta} \qquad 3.4.21$$

$$\omega_r^o = -\frac{u_r^o}{2f\left[1+\gamma^2\right]^{3/2}} - \frac{1}{2f\sqrt{1+\gamma^2}} \frac{\partial w}{\partial r}, \qquad 3.4.22$$

$$\omega_\theta^o = -\frac{u_\theta^o}{2f\sqrt{1+\gamma^2}} - \frac{1}{2f\gamma} \frac{\partial w}{\partial \theta}, \qquad 3.4.23$$

$$K_r = \frac{1}{2f\sqrt{1+\gamma^2}} \frac{\partial \omega_r^o}{\partial \gamma}, \qquad 3.4.24$$

$$K_\theta = \frac{\omega_r^o}{2f\gamma\sqrt{1+\gamma^2}} + \frac{1}{2f\gamma} \frac{\partial \omega_\theta^o}{\partial \theta} \qquad 3.4.25$$

$$K_{r\theta} = \frac{1}{2f\sqrt{1+\gamma^2}} \frac{\partial \omega_\theta^o}{\partial \gamma} - \frac{\omega_\theta^o}{2f\gamma\sqrt{1+\gamma^2}} + \frac{1}{2f\gamma} \frac{\partial \omega_r^o}{\partial \theta}. \qquad 3.4.26$$

## 3.5 STRAIN-DISPLACEMENT RELATIONS: CARTESIAN PARAMETERS

We will summarize in this section the strain-displacement relations for the paraboloid as described by the Cartesian parameters x and y.

$$\gamma_{11}^{o} = \left(\frac{\partial u^1}{\partial x} + \frac{x u^1}{1+(x)^2+(y)^2}\right) 4f^2\left[1+(x)^2\right] + \left(\frac{\partial u^2}{\partial x} + \frac{y u^1}{1+(x)^2+(y)^2}\right) 4f^2 xy$$
$$- \frac{2fw}{\sqrt{1+(x)^2+(y)^2}}, \qquad 3.5.1$$

$$\gamma_{22}^{o} = \left(\frac{\partial u^2}{\partial y} + \frac{y u^2}{1+(x)^2+(y)^2}\right. \; 4f^2\left[1+(y)^2\right] + \left(\frac{du^1}{dy} + \frac{x u^2}{1+(x)^2+(y)^2}\right) 4f^2 xy$$
$$- \frac{2fw}{\sqrt{1+(x)^2+(y)^2}}, \qquad 3.5.2$$

$$\gamma_{12}^{o} = \frac{1}{2}\left\{\left(\frac{\partial u^1}{\partial y} + \frac{x u^2}{1+(x)^2+(y)^2}\right) 4f^2\left[1+(x)^2\right] + \left(\frac{\partial u^2}{\partial x} + \frac{y u^1}{1+(x)^2+(y)^2}\right) 4f^2\left[1+(y)^2\right]\right.$$
$$+ \left(\frac{\partial u^1}{\partial x} + \frac{x u^1}{1+(x)^2+(y)^2} + \frac{\partial u^2}{\partial y} + \frac{y u^2}{1+(x)^2+(y)^2}\right. \; 4f^2 xy$$
$$\left. - 2b_{12}\, w\right\}, \qquad 3.5.3$$

$$k_{11} = \left(\frac{\partial \omega^1}{\partial x} + \frac{x \omega^1}{1+(x)^2+(y)^2}\right) 4f^2\left[1+(x)^2\right] + \left(\frac{\partial \omega^2}{\partial x} + \frac{y \omega^1}{1+(x)^2+(y)^2}\right) 4f^2 xy, \qquad 3.5.4$$

$$k_{22} = \left(\frac{\partial \omega^2}{\partial y} + \frac{y \omega^2}{1+(x)^2+(y)^2}\right) 4f^2\left[1+(x)^2\right] + \left(\frac{\partial \omega^1}{\partial y} + \frac{x \omega^2}{1+(x)^2+(y)^2}\right) 4f^2 xy, \qquad 3.5.5$$

$$k_{12} = \frac{1}{2}\left\{\left(\frac{\partial \omega^1}{\partial y} + \frac{x \omega^2}{1+(x)^2+(y)^2}\right) 4f^2\left[1+(x)^2\right] + \left(\frac{\partial \omega^2}{\partial x} + \frac{y \omega^1}{1+(x)^2+(y)^2}\right) 4f^2\left[1+(y)^2\right]\right.$$
$$\left. + \left(\frac{\partial \omega^1}{\partial x} + \frac{x \omega^1}{1+(x)^2+(y)^2} + \frac{\partial \omega^2}{\partial y} + \frac{y \omega^2}{1+(x)^2+(y)^2}\right) 4f^2 xy\right\}, \qquad 3.5.6$$

$$\omega^1 = -\frac{[1+(y)^2]\,u^1}{2f[1+(x)^2+(y)^2]^{3/2}} - \frac{xyu^2}{2f[1+(x)^2+(y)^2]^{3/2}} \qquad 3.5.7$$

$$- \frac{1+(y)^2}{4f^2[1+(x)^2+(y)^2]}\frac{\partial w}{\partial x} - \frac{xy}{4f^2[1+(x)^2+(y)^2}\frac{\partial w}{\partial y} .$$

$$\omega^2 = -\frac{[1+(x)^2]\,u^2}{2f[1+(x)^2+(y)^2]^{3/2}} - \frac{xyu^1}{2f[1+(x)^2+(y)^2]^{3/2}}$$

$$- \frac{1+(x)^2}{4f^2[1+(x)^2+(y)^2]}\frac{\partial w}{\partial y} - \frac{xy}{4f^2[1+(x)^2+(y)^2]}\frac{\partial w}{\partial x} . \qquad 3.5.8$$

In this case of the rectangular parameters, the physical components of the displacements will be taken with respect to the covariant base vectors, $\bar{a}_\alpha$ . Thus we will use

$$u_1^o = \sqrt{a_{11}}\quad u^1 = 2f\sqrt{1+(x)^2}\; u^1, \qquad 3.5.9$$

$$u_2^o = \sqrt{a_{22}}\quad u^2 = 2f\sqrt{1+(y)^2}\; u^2, \qquad 3.5.10$$

$$\omega_1^o = \sqrt{a_{11}}\quad \omega^1 = 2f\sqrt{1+(x)^2}\; \omega^1, \qquad 3.5.11$$

$$\omega_2^o = \sqrt{a_{22}} \quad \omega^2 = 2f\sqrt{1+(\gamma)^2} \quad \omega^2 \ . \qquad 3.5.12$$

IV ANALYSIS OF STRESSES

4.1 THE STRESS TENSOR

The stress tensor in three-dimensional space is defined on the basis of the equilibrium of a tetrahedron which is carved out of the deformed body (see figure 4.1.1).

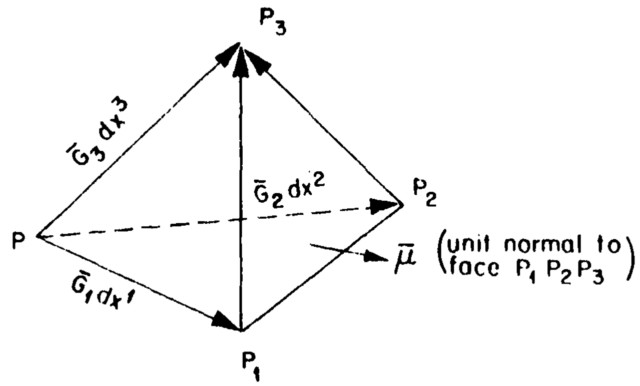


Figure 4.1.1

In the limit, three deges of the tetrahedron can be considered to be formed by the vectors $\bar{G}_1 dx^1$, $\bar{G}_2 dx^2$, and $\bar{G}_3 dx^3$, where the $\bar{G}_n$ are the base vectors of the deformed body. The fourth face of the tetrahedron is located by an outwardly directed unit normal vector, $\bar{\mu}$ , which can be expressed as

$$\bar{\mu} = \mu^m \bar{G}_m = \mu_m \bar{G}^m. \qquad 4.1.1$$

The action of the rest of the body on the tetrahedron is represented by the stress vectors shown in figure 4.1.2:

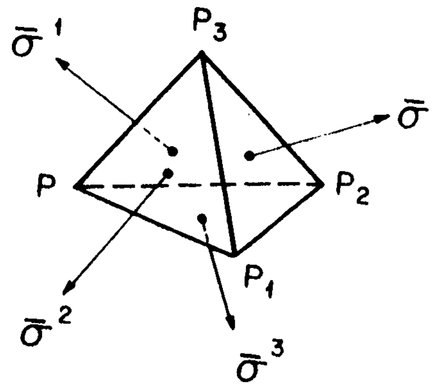


Figure 4.1.2

In this figure we have the following:

$\bar{\sigma}$ = stress vector (force per unit area) acting on face dA

$\bar{\sigma}^m$ = stress vector (force per unit area) acting on dAm

$dA$ = one-half of the area of face $P_1P_2P_3$

$dA_1$ = one-half of the area of face $P\ P_2P_3$ (coordinate surface $\xi^1$ = constant)

$dA_2$ = one-half of the area of face $P\ P_1P_3$ (coordinate surface $\xi^2$ = constant)

$dA_3$ = one half of the area of face $P\ P_1P_2$ (coordinate surface $\xi^3$ = constant)

The equilibrium of the tetrahedron requires

$$\bar{\sigma}\, dA - \bar{\sigma}^m\, dA_m = 0 \qquad 4.1.2$$

and since

$$dA_m = n_m \sqrt{G^{mm}}\, dA \qquad \text{(No Sum)} \qquad 4.1.3$$

the equilibrium equation becomes

$$\bar{\sigma} = \sum_{m=1}^{3} \bar{\sigma}^m \sqrt{G^{mm}}\; n_m \qquad 4.1.4$$

This last relation forms the basis on which the stress tensor is defined. We write

$$\bar{\sigma}^m \sqrt{G^{mm}} = \sum_{n=1}^{3} \tau^{mn}\; \bar{G}_n \qquad \text{(No Sum on m)} \qquad 4.1.5$$

It can be shown that the stress tensor is symmetric:

$$\tau^{mn} = \tau^{nm}. \qquad 4.1.6$$

## 4.2 THE FORCE AND MOMENT RESULTANTS

Generally, a shell structure is one in which dimensions in the two coordinate directions $\xi^1$ and $\xi^2$ are large compared to the dimension in direction $\zeta$ which is normal to $\xi^1$ and $\xi^2$ . Additionally, the bounding surfaces $\zeta = \pm \frac{h}{2}$ are usually acted upon by loads of a magnitude which cause the surface stresses at the bounding surfact to be negligibly small in comparison to the internal stresses. It thus becomes more convenient to work with stress resultants which are the stress effects integrated across the thickness of the shell. This procedure, it may be recalled, is one which is used in the theory of beams and plates.

The stress resultants are defined relative to a shell volume element which extends over the total thickness of the shell (see figure 4.2.1).

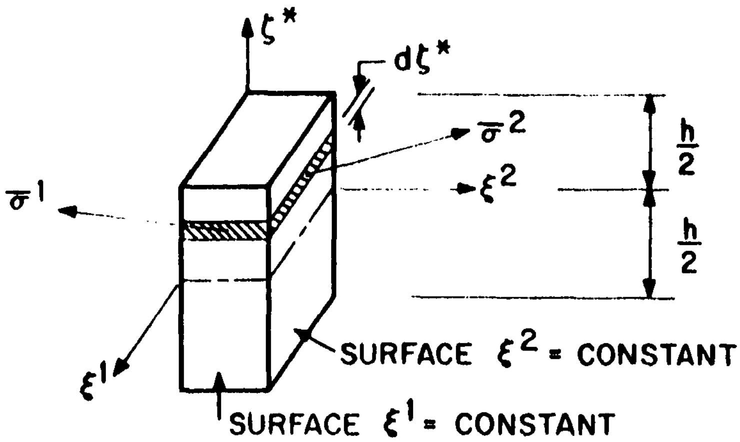


Figure 4.2.1

There are shown in figure 4.2.1 the stress vectors $\bar{\sigma}^1$ and $\bar{\sigma}^2$ which act on elements of area $\sqrt{G_{22}}\, d\xi^2 d\zeta^*$ and $\sqrt{G_{11}}\, d\xi^1 d\zeta^*$ respectively (see figure 4.2.2).

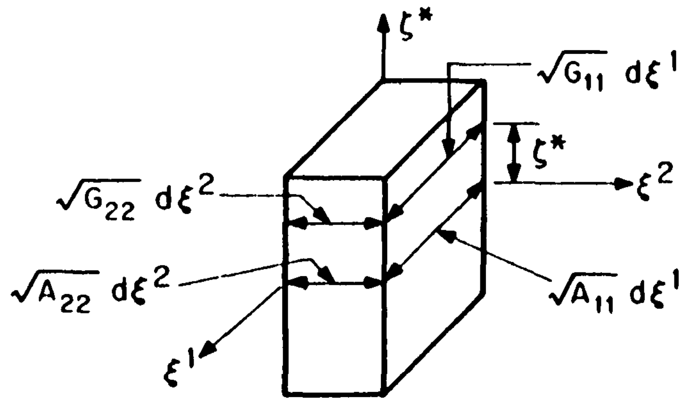


Figure 4.2.2

The stress vectors, which are a function of $\zeta^{\star}$, are replaced by stress resultants. Thus, figure 4.2.1 which shows the stress vectors and the shell volume element is replaced by figure 4.2.3 which shows the middle surface of the volume element and the stress resultants.

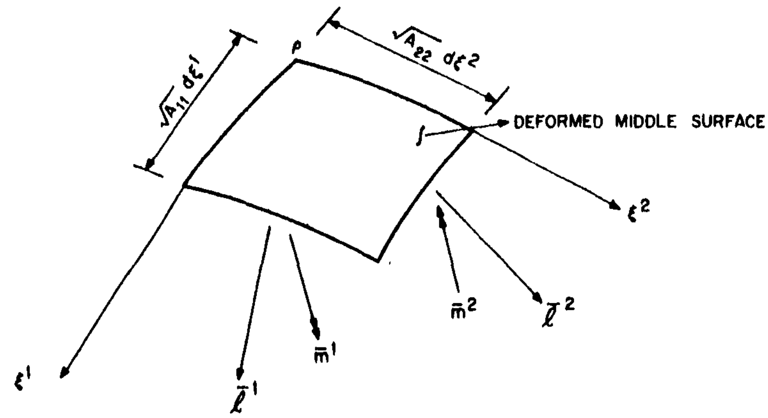


Figure 4.2.3

There are two kinds of stress resultants: $\ell^1$ and $\ell^2$ which will be called force resultants; and $\bar{m}^1$ and $\bar{m}^2$ which are called moment resultants. It will be shown that the moment resultants are surface vectors, i.e., they lie in the tangent plane at point p. The force resultants, however, are space vectors.

The definitions for the stress resultants are in reality equations which characterize the statical equivalence of the stress resultants to the stress vectors:

$$\bar{\ell}^1\sqrt{A_{22}}\ d\xi^2 = \int_{-\frac{h^*}{2}}^{\frac{h^*}{2}} \bar{\sigma}^1\sqrt{G_{22}}\ d\xi^2\, d\zeta^*, \qquad 4.2.1$$

$$\bar{\ell}^2\sqrt{A_{11}}\ d\xi^1 = \int_{-\frac{h^*}{2}}^{\frac{h^*}{2}} \bar{\sigma}\sqrt{G_{11}}\ d\xi^1\, d\zeta^*, \qquad 4.2.2$$

$$\bar{m}^1\sqrt{A_{22}}\ d\xi^2 = \int_{-\frac{h^*}{2}}^{\frac{h^*}{2}} \zeta^*\bar{N}\times\bar{\sigma}^1\sqrt{G_{22}}\ d\xi^2 d\zeta^*. \qquad 4.2.3$$

$$\bar{m}^2\sqrt{A_{11}}\ d\xi^1 = \int_{-\frac{h}{2}}^{\frac{h}{2}^*} \zeta^*\bar{N}\times\bar{\sigma}^2\sqrt{G_{11}}\ d\xi^1 d\zeta^* \qquad 4.2.4$$

It is useful at this stage to introduce the stress tensor by means of the equations (see equation 4.1.5),

$$\bar{\sigma}^1 = \frac{1}{\sqrt{G^{11}}}\left[\tau^{11}\bar{G}_1 + \tau^{12}\bar{G}_2 + \tau^{13}\bar{N}\right], \qquad 4.2.5$$

$$\bar{\sigma}^2\ \frac{1}{\sqrt{G^{22}}}\left[\tau^{21}\bar{G}_1 + \tau^{22}\bar{G}_2 + \tau^{23}\bar{N}\right]. \qquad 4.2.6$$

We will define the force resultant tensor and the moment resultant tensor by means of the equations

$$\bar{\ell}^1 \sqrt{A^{11}} = \ell^{11} \bar{A}_1 + \ell^{12} \bar{A}_2 + q^1 \bar{N}, \tag{4.2.7}$$

$$\bar{\ell}^2 \sqrt{A^{22}} = \ell^{21} \bar{A}_1 + \ell^{22} \bar{A}_2 + q^2 \bar{N}, \tag{4.2.8}$$

$$\bar{m}^1 \sqrt{A^{11}} = \sqrt{A}\ m^{11} \bar{A}^2 - \sqrt{A}\ m^{12} \bar{A}^1, \tag{4.2.9}$$

$$\bar{m}^2 \sqrt{A^{22}} = \sqrt{A}\ m^{21} \bar{A}^2 - \sqrt{A}\ m^{22} \bar{A}^2 . \tag{4.2.10}$$

It can be shown that the quantities $\ell^{\alpha\beta}$ and $m^{\alpha\beta}$ are surface tensors of the contravariant type.

If equations 4.2.5 and 4.2.6 are substituted into equations 4.2.1 to 4.2.4 and the results compared with equation 4.2.7 to 4.2.10, there will emerge the following relations for the stress and moment resultant tensors:

$$\ell^{11} = \int_{-\frac{h^\star}{2}}^{\frac{h^\star}{2}} \sqrt{\frac{G}{A}} \left[\tau^{11}(1-\zeta^\star B_1^1) - \tau^{12}\zeta^\star B_2^1\right] d\zeta^\star, \qquad 4.2.11$$

$$\ell^{12} = \int_{-\frac{h^\star}{2}}^{\frac{h^\star}{2}} \sqrt{\frac{G}{A}} \left[-\tau^{11}\zeta^\star B_1^2 + \tau^{12}\left(1-\zeta^\star B_2^2\right)\right] d\zeta^\star, \qquad 4.2.12$$

$$\ell^{21} = \int_{-\frac{h^\star}{2}}^{\frac{h^\star}{2}} \sqrt{\frac{G}{A}} \left[\tau^{21}(1-\zeta^\star B_1^1) - \tau^{22}\zeta^\star B_2^1\right] d\zeta^\star, \qquad 4.2.13$$

$$\ell^{22} = \int_{-\frac{h^\star}{2}}^{\frac{h^\star}{2}} \sqrt{\frac{G}{A}} \left[-\tau^{22}\zeta^\star B_1^2 + \tau^{22}\left(1-\zeta^\star B_2^2\right)\right] d\zeta^\star, \qquad 4.2.14$$

$$q^1 = \int_{-\frac{h}{2}}^{\frac{h^\star}{2}} \sqrt{\frac{G}{A}} \; \tau^{13} \, d\zeta^\star, \qquad 4.2.15$$

$$q^2 = \int_{-\frac{h^\star}{2}}^{\frac{h^\star}{2}} \sqrt{\frac{G}{A}} \; \tau^{23} \, d\zeta^\star, \qquad 4.2.16$$

$$m^{11} = \int_{-\frac{h^\star}{2}}^{\frac{h^\star}{2}} \sqrt{\frac{G}{A}} \left[\tau^{11}\left(1-\zeta^\star B_1^1\right) - \tau^{12}\zeta^\star B_2^1\right] \zeta^\star d\zeta^\star, \qquad 4.2.17$$

$$m^{12} = \int_{-\frac{h^\star}{2}}^{\frac{h^\star}{2}} \sqrt{\frac{G}{A}} \left[ -\tau^{11} \zeta^\star B_1^2 + \tau^{12} \left( 1 - \zeta^\star B_2^2 \right) \right] \zeta^\star d\zeta^\star, \qquad 4.2.18$$

$$m^{21} = \int_{-\frac{h^\star}{2}}^{\frac{h^\star}{2}} \sqrt{\frac{G}{A}} \left[ \tau^{21} \left( 1 - \zeta^\star B_1^1 \right) - \tau^{22} \zeta^\star B_2^1 \right] \zeta^\star d\zeta^\star, \qquad 4.2.19$$

$$m^{22} = \int_{-\frac{h^\star}{2}}^{\frac{h^\star}{2}} \sqrt{\frac{G}{A}} \left[ -\tau^{21} \zeta^\star B_1^2 + \tau^{22} \left( 1 - \zeta^\star B_2^2 \right) \right] \zeta^\star d\zeta^\star. \qquad 4.2.20$$

Note that the equations 4.2.11 to 4.2.16, which may be termed the exact definitions, define tensors which are not symmetric.

If the deformed shape of the shell is assumed to be such that the radius of curvature of the deformed middle surface is still large in comparison to the thickness, $h^\star$, then the terms $\zeta^\star B_\beta^\alpha$ will be small in comparison to unity and hence can be neglected in equations 4.2.11 to 4.2.20. Additionally, there will be a negligible difference between the length of $h$ and $h^\star$ for the case of small strain. With these restrictions, the components of the force-resultant and moment resultant tensors assume the simpler forms shown below:

$$\ell^{11} = \int_{-\frac{h}{2}}^{\frac{h}{2}} \tau^{11} \, d\zeta , \qquad 4.2.21$$

$$\ell^{12} = \ell^{21} = \int_{-\frac{h}{2}}^{\frac{h}{2}} \tau^{12} \, d\zeta , \qquad 4.2.22$$

$$\ell^{22} = \int_{-\frac{h}{2}}^{\frac{h}{2}} \tau^{22} \, d\zeta , \qquad 4.2.23$$

$$q^{1} = \int_{-\frac{h}{2}}^{\frac{h}{2}} \tau^{13} \, d\zeta , \qquad 4.2.24$$

$$q^{2} = \int_{-\frac{h}{2}}^{\frac{h}{2}} \tau^{23} \, d\zeta , \qquad 4.2.25$$

$$m^{11} = \int_{-\frac{h}{2}}^{\frac{h}{2}} \tau^{11} \zeta \, d\zeta , \qquad 4.2.26$$

$$m^{12} = m^{21} = \int_{-\frac{h}{2}}^{\frac{h}{2}} \tau^{12} \zeta \, d\zeta , \qquad 4.2.27$$

$$m^{22} = \int_{-\frac{h}{2}}^{\frac{h}{2}} \tau^{22}\, \zeta\, d\zeta . \qquad 4.2.28$$

It should be observed that the tensors $m^{\alpha\beta}$ and $\ell^{\alpha\beta}$ are symmetrical tensors for a thin shell.

## 4.3 THE EQUATIONS OF EQUILIBRIUM

It will be found convenient to introduce the quasi-force vectors $\bar{L}^{\alpha}$ defined by setting

$$\bar{L}^{1}\, d\xi^{2}\, \bar{\ell}^{1} \sqrt{A_{22}}\; d\xi^{2}, \qquad 4.3.1$$

$$\bar{L}^{2}\, d\xi^{1} = \bar{\ell}^{2} \sqrt{A_{11}}\; d\xi^{1}. \qquad 4.3.2$$

The quantities $\bar{L}^{1} d\xi^{2}$ and $\bar{L}^{2} d\xi^{1}$ have the dimensions of a force, and by introducing equations 4.2.7 and 4.2.8, it is seen that

$$\bar{L}^{\sigma} = \sqrt{A}\, \left\{ \ell^{\sigma\alpha} \bar{A}_{\alpha} + q^{\sigma} \bar{N} \right\}. \qquad 4.3.3$$

Similarly, we introduce the quasi-moment vectors $\bar{M}^{\alpha}$ by setting

$$\bar{M}^{1} d\xi^{2} = \bar{m}^{1}\sqrt{A_{22}}\; d\xi^{2}, \qquad 4.3.4$$

$$\bar{M}^{2} d\xi^{1} = \bar{m}^{2}\sqrt{A_{11}}\; d\xi^{1} \qquad 4.3.5$$

By substituting equations 4.2.9 and 4.2.10, there results

$$\bar{M}^{\sigma} = \sqrt{A}\;\varepsilon_{\alpha\beta}\; m^{\sigma\alpha}\bar{A}^{\beta} \qquad 4.3.6$$

where $\varepsilon_{\alpha\beta}$ is the covariant permutation surface tensor.

The force equilibrium of the shell element of volume can be obtained by examining figure 4.3.1, which sketches the force vectors which are acting at the deformed middle surface.

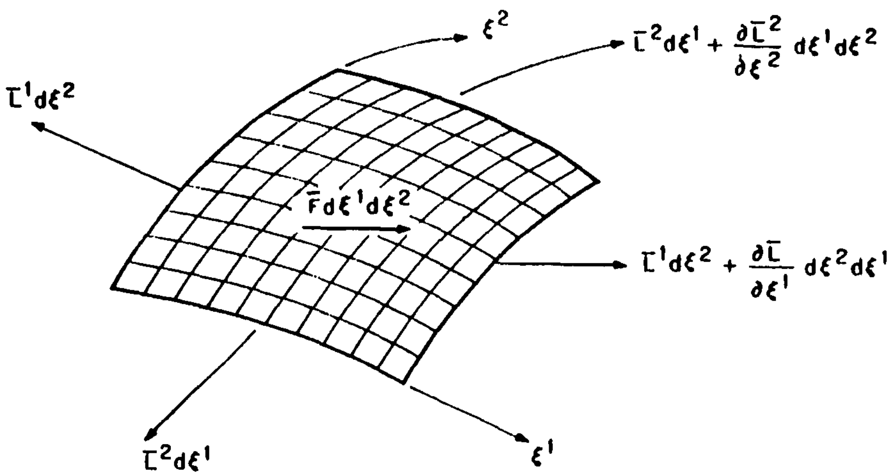


Figure 4.3.1

The vector equation of equilibrium is simply

$$\left(\frac{\partial \bar{L}^1}{\partial \xi^1} + \frac{\partial \bar{L}^2}{\partial \xi^2} + \bar{F}\right) d\xi^1 \, d\xi^2 = 0 \qquad 4.3.7$$

where $\bar{F} \, d\xi^1 d\xi^2$ is a body force vector with components given by

$$\bar{F} = \sqrt{A} \, \left(F^{\alpha} \bar{A}_{\alpha} + F^3 \, \bar{N}\right). \qquad 4.3.8$$

It is easily demonstrated that equation 4.3.7 has the following form if the stress resultant tensor is introduced by means of equation 4.3.3.

$$\left[\ell^{\beta\alpha}\Big|_{\beta} - q^{\beta} B_{\beta}^{\alpha} + F^{\alpha}\right]\sqrt{A}\ \bar{A}_{\alpha}$$

$$+\left[q^{\beta}\Big|_{\beta} + \ell^{\beta\alpha} B_{\alpha\beta} + F^{3}\right]\sqrt{A}\ \bar{N} = 0. \qquad 4.3.9$$

The vertical slash before a subscript indicates surface covariant differentiation in the metric of the deformed middle surface.

The moment equilibrium of the shell element of volume requires the examination of figure 4.3.2, which sketches the moment vectors acting on the deformed middle surface, and also figure 4.3.1.

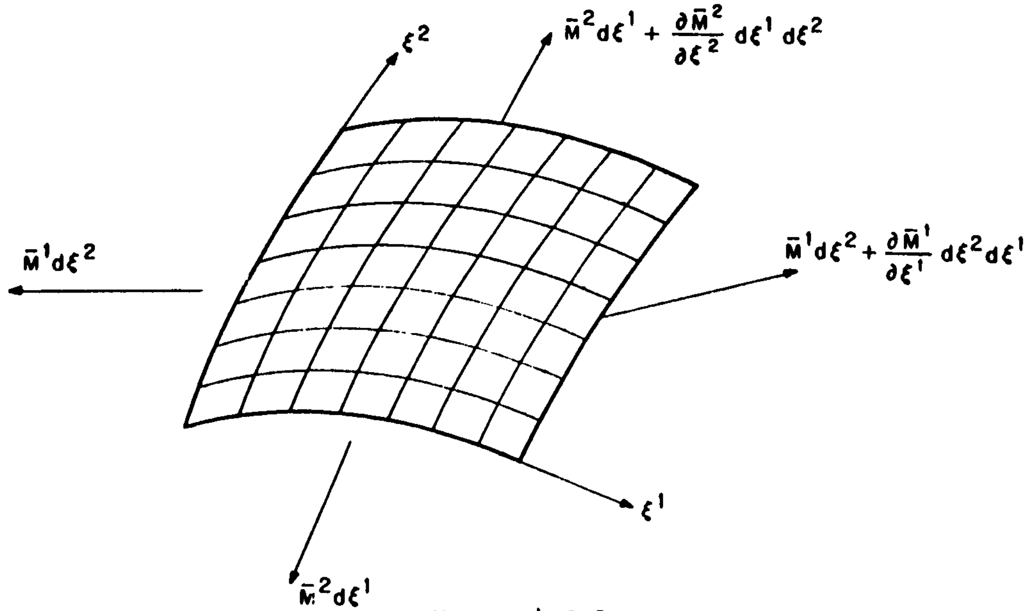


Figure 4.3.2

The moments depicted in figure 4.3.2 are moments of the stresses about the $\xi^1$ and $\xi^2$ axes. In addition, there are moments about the $\zeta^*$ axis caused by the force vectors shown in figure 4.3.1. By properly summing up all of these moments, the following vector equation of moment equilibrium is obtained:

$$\left(\frac{\partial \bar{M}^{\alpha}}{\partial \xi^{\alpha}} + \bar{A}_{\alpha} \times \bar{L}^{\alpha}\right) d\xi^1 \, d\xi^2 = 0. \qquad 4.3.10$$

There are terms of higher order such as that contributed by the body force $\bar{F} d\xi^1 d\xi^2$ which have been neglected.

By introducing equation 4.3.6, the moment equilibrium equation becomes

$$\left[m^{\alpha\sigma}\big|_{\alpha} - q^{\sigma}\right] \varepsilon_{\sigma\tau} \sqrt{A}\, \bar{A}^{\tau} + \left[B^{\gamma}_{\alpha}\, m^{\alpha\sigma} + \ell^{\sigma\gamma}\right] \varepsilon_{\sigma\gamma} \sqrt{A}\, \bar{N} = 0. \qquad 4.3.11$$

The scalar equations of force equilibrium in the $\bar{A}^1$, $\bar{A}^2$, and $\bar{N}$ directions are obtained by forming the scalar product of equation 4.3.9 with the contravariant base vectors $\bar{A}^1$, $\bar{A}^2$, and $\bar{N}$

respectively:

$$\ell^{\beta\alpha}\big|_{\beta} - q^{\beta} B^{\alpha}_{\beta} + F^{\alpha} = 0, \qquad 4.3.12$$

$$q^{\beta}\big|_{\beta} + \ell^{\beta\alpha} B_{\alpha\beta} + F^{3} = 0. \qquad 4.3.13$$

The scalar equations of moment equilibrium in the $\bar{A}_1$, $\bar{A}_2$ directions are obtained by forming the scalar product of equation 4.3.11 with the covariant base vectors $\bar{A}_1$, and $\bar{A}_2$:

$$\left(m^{\alpha\sigma}\big|_{\alpha} - q^{\sigma}\right)\varepsilon_{\sigma\tau} = 0. \qquad 4.3.14$$

It can be shown that the scalar equation of moment equilibrium about the $\bar{N}$ direction,

$$\left(B^{\gamma}_{\alpha}\ m^{\alpha\sigma} + \ell^{\sigma\gamma}\right)\ \varepsilon_{\sigma\gamma} = 0 \qquad 4.3.15$$

is identically satisfied because the strain tensor, $\tau^{mn}$, is symmetric.

## 4.4 THE EQUILIBRIUM EQUATIONS FOR INFINITESIMAL STRAINS

If the strains are infinitesimal, then insofar as the stresses are concerned, the geometry of the deformed shell is indistinguishable from the geometry of the undeformed shell, Thus, the first and second fundamental tensors of the deformed middle surface appearing in the equilibrium equations of section 4.3 can be replaced by the similar tensors of the undeformed middle surface. The equations of equilibrium under these circumstances will be written as follows:

$$\ell^{\beta\alpha}_{,\beta} - q^{\beta} b^{\alpha}_{\beta} F^{\alpha} = 0, \qquad 4.4.1$$

$$q^{\beta}_{,\beta} + \ell^{\beta\alpha} b_{\alpha\beta} + F^{3} = 0, \qquad 4.4.2$$

$$m^{\alpha\sigma}_{,\alpha} - q^{\sigma} = 0. \qquad 4.4.3$$

The comma now signifies surface corvariant differentiation with respect to the metric tensor of the undeformed middle surface.

### 4.4.1 POLAR PARAMETERS

The equations 4.4.1, 4.4.2, and 4.4.3 assume the following forms if the polar parameters $\gamma$ and $\theta$ of section 2.1 are used.

$$\frac{\partial \ell^{11}}{\partial \gamma} + \frac{\partial \ell^{21}}{\partial \theta} + \left[\frac{2\gamma}{1+(\gamma)^2} + \frac{1}{\gamma}\right] \ell^{11} - \left(\frac{\gamma}{1+(\gamma)^2}\right) \ell^{22} - \frac{q^1}{2f[1+(\gamma)^2]^{3/2}} + F^1 = 0, \qquad 4.4.1.1$$

$$\frac{\partial \ell^{12}}{\partial \gamma} + \frac{\partial \ell^{22}}{\partial \theta} + \left[\frac{3}{\gamma} + \frac{\gamma}{1+(\gamma)^2}\right] \ell^{12} - \frac{q^1}{2f[1+(\gamma)^2]^{1/2}} + F^2 = 0, \qquad 4.4.1.2$$

$$\frac{\partial q^1}{\partial \gamma} + \frac{\partial q^2}{\partial \gamma} + \left[\frac{\gamma}{1+(\gamma)^2} + \frac{1}{\gamma}\right] q^1 + \frac{2f}{[1+(\gamma)^2]^{1/2}} \ell^{11} + \frac{2f\gamma^2}{[1+(\gamma)^2]^{1/2}} \ell^{22} + F^3 = 0, \qquad 4.4.1.3$$

$$\frac{\partial m^{11}}{\partial \gamma} + \frac{\partial m^{21}}{\partial \theta} + \left[\frac{2\gamma}{1+(\gamma)^2} + \frac{1}{\gamma}\right] m^{11} - \frac{\gamma}{1+(\gamma)^2} m^{22} - q^1 = 0, \qquad 4.4.1.4$$

$$\frac{\partial m^{12}}{\partial \gamma} + \frac{\partial m^{22}}{\partial \theta} + \left[\frac{3}{\gamma} + \frac{\gamma}{1+(\gamma)^2}\right] m^{12} - q^2 = 0. \qquad 4.4.1.5$$

The physical components of the force and moment resultant tensors are, since the polar parameters are also the lines of curvature, given by

$$N_r \equiv N_{11} = \ell^{11} a_{11} = 4f^2\left[1+(\gamma)^2\right] \ell^{11}, \qquad 4.4.1.6$$

$$N_\theta \equiv N_{22} = \ell^{22} a_{22} = 4f^2 (\gamma)^2 \ell^{22}, \qquad 4.4.1.7$$

$$N_{r\theta} \equiv N_{12} = N_{21} = \ell^{12} \sqrt{a_{11} a_{22}} = 4f^2 \gamma \left[1+(\gamma)^2\right]^{1/2} \ell^{12}, \qquad 4.4.1.8$$

$$M_r \equiv M_{11} = m^{11} a_{11} = 4f^2\left[1+(\gamma)^2\right] m^{11}, \qquad 4.4.1.9$$

$$M_\theta \equiv M_{22} = m^{22} a_{22} = 4f^2 (\gamma)^2 m^{22}, \qquad 4.4.1.10$$

$$M_{r\theta} = M_{12} = M_{21} = 4f^2 \gamma \left[1+(\gamma)^2\right]^{1/2} m^{12}, \qquad 4.4.1.11$$

$$Q_r = Q_1 = \sqrt{a_{11}}\ q^1 = 2f\sqrt{1+(\gamma)^2}\ q^1, \qquad 4.4.1.12$$

$$Q_\theta = Q_2 = \sqrt{a_{22}}\ q^2 = 2f\gamma q^2. \qquad 4.4.1.13$$

If the physical components are used, the equations of equilibrium can be arranged into the following form:

$$\gamma \frac{\partial N_r}{\partial \gamma} + \sqrt{1+(\gamma)^2}\ \frac{\partial N_{r\theta}}{\partial \theta} + N_r - N_\theta - \frac{\gamma}{1+(\gamma)^2}\ Q_r + 2f\gamma\sqrt{1+(\gamma)^2}\ p_r = 0, \qquad 4.4.1.14$$

$$\gamma \frac{\partial N_{r\theta}}{\partial \gamma} + \sqrt{1+(\gamma)^2}\ \frac{\partial N_\theta}{\partial \theta} + 2N_{r\theta} - \gamma Q_\theta + 2f\gamma\sqrt{1+(\gamma)^2}\ p_\theta = 0, \qquad 4.4.1.15$$

$$\gamma \frac{\partial Q_r}{\partial \gamma} + \sqrt{1+(\gamma)^2}\ \frac{\partial Q_\theta}{\partial \theta} + \frac{\gamma}{1+(\gamma)^2}\ N_r + \gamma N_\theta + Q_r + 2f\gamma\sqrt{1+(\gamma)^2}\ p_3 = 0, \qquad 4.4.1.16$$

$$\gamma \frac{\partial M_r}{\partial \gamma} + \sqrt{1+(\gamma)^2}\ \frac{\partial M_{r\theta}}{\partial \gamma} + M_r - M_\theta - 2f\gamma\sqrt{1+(\gamma)^2}\, Q_r = 0, \qquad 4.4.1.17$$

$$\gamma \frac{\partial M_{r\theta}}{\partial \gamma} + \sqrt{1+(\gamma)^2}\ \frac{\partial M_\theta}{\partial \theta} + 2M_{r\theta} - 2f\gamma\sqrt{1+(\gamma)^2}\ Q_\theta = 0 \qquad 4.4.1.18$$

where the body force intensity vector (see equation 4.3.8) is given by

$$\bar{p} = p_r \frac{\bar{a}_1}{\sqrt{a_{11}}} + p_\theta \frac{\bar{a}_2}{\sqrt{a_{22}}} + p_n \bar{n} \, . \qquad 4.4.1.19$$

It should be observed that $\frac{\bar{a}_1}{\sqrt{a_{11}}}$ and $\frac{\bar{a}_2}{\sqrt{a_{22}}}$ are unit vectors.

## 4.4.2 CARTESIAN PARAMETERS

The equations 4.4.1, 4.4.2, and 4.4.3 assume the following forms if the cartesian parameters x and y of section 2.3 are used.

$$\frac{\partial \ell^{11}}{\partial x} + \frac{\partial \ell^{21}}{\partial y} + \frac{32f^4}{a} x \ell^{11} + \frac{16f^4}{a} y \ell^{12} + \frac{16f^4}{a} x \ell^{22} - \frac{8a_{22}f^3}{a\sqrt{a}} q^1 + \frac{8a_{12}f^3}{a\sqrt{a}} q^2 + F^1 = 0, \qquad 4.4.2.1$$

$$\frac{\partial \ell^{21}}{\partial x} + \frac{\partial \ell^{22}}{\partial y} + \frac{32f^4}{a} y \ell^{22} + \frac{16f^4}{a} x \ell^{21} + \frac{16f^4}{a} y \ell^{11} + \frac{8a_{12}f^3}{a\sqrt{a}} q^1 - \frac{8a_{11}f^3}{a\sqrt{a}} q^2 + F^2 = 0, \qquad 4.4.2.2$$

$$\frac{\partial q^1}{\partial x} + \frac{\partial q^2}{\partial y} + \frac{16f^4}{a} x q^1 + \frac{16f^4}{a} y q^2 + \frac{8f^3}{\sqrt{a}} \ell^{11} + \frac{8f^3}{\sqrt{a}} \ell^{22} + F^3 = 0, \qquad 4.4.2.3$$

$$\frac{\partial m^{11}}{\partial x} + \frac{\partial m^{21}}{\partial y} + \frac{32f^4}{a} x m^{11} + \frac{16f^4}{a} y m^{12} + \frac{16f^4}{a} x m^{22} - q^1 = 0, \qquad 4.4.2.4$$

$$\frac{\partial m^{12}}{\partial x} + \frac{\partial m^{22}}{\partial y} + \frac{32f^4}{a} ym^{22} + \frac{16f^4}{a} xm^{21} + \frac{16f^4}{a} ym^{11} - q^2 = 0. \qquad 4.4.2.5$$

It should be remembered that $f$ is the focal length and hence $f^4$ and $f^3$ represent powers of $f$. The components of $a_{\alpha\beta}$ are detailed in section 2.3 as is $a$.

The physical components of the force and moment resultant tensors are given by

$$N_x = N_{11} = \sqrt{a}\sqrt{\frac{a_{11}}{a_{22}}}\,\ell^{11}, \qquad 4.4.2.6$$

$$N_{xy} = N_{12} = \sqrt{a}\,\ell^{12}, \qquad 4.4.2.7$$

$$N_y = N_{22} = \sqrt{\frac{a_{22}}{a_{11}}}\,\ell^{22}\sqrt{a}, \qquad 4.4.2.8$$

$$Q_x = Q_1 = \sqrt{a_{11}}\,q^1, \qquad 4.4.2.9$$

$$Q_y = q_2 = \sqrt{a_{22}} \quad q^2, \tag{4.4.2.10}$$

$$M_x = M_{11} = \sqrt{a} \sqrt{\frac{a_{11}}{a_{22}}} \; \ell^{11}, \tag{4.4.2.11}$$

$$M_{xy} = M_{12} = \sqrt{a} \; m^{12}, \tag{4.4.2.12}$$

$$M_y = M_{22} = \sqrt{a} \sqrt{\frac{a_{22}}{a_{11}}} \; \ell_{22}, \tag{4.4.2.13}$$

$$p_x = p_1 = \sqrt{a_{11}} \; F^1, \tag{4.4.2.14}$$

$$p_y = p_2 = \sqrt{a_{22}} \; F^2, \tag{4.4.2.15}$$

$$p_\zeta = p_3 = F^3. \tag{4.4.2.16}$$

The equations of equilibrium in terms of the physical components of the force and moment resultants are listed below:

$$\frac{\partial N_x}{\partial x}+\left[\frac{16f^4}{a}-\frac{4f^2}{a_{11}}\right] x N_x+\sqrt{\frac{a_{11}}{a_{22}}}\ \frac{\partial N_{xy}}{\partial y}+\frac{a_{11}}{a_{22}}\ \frac{16f^4}{a}\ x\ N_y$$

$$-\frac{8\sqrt{a_{22}}\,f^3}{a}\ Q_x+\frac{8\sqrt{a_{11}}\ a_{12}f^3}{a\ a_{22}}\ Q_y+\sqrt{\frac{a}{a_{22}}}\ p_x=0, \qquad 4.4.2.17$$

$$\frac{\partial N_y}{\partial y}+\left[\frac{16f^4}{a}-\frac{4f^2}{a_{22}}\right] y N_y+\sqrt{\frac{a_{22}}{a_{11}}}\ \frac{\partial N_{xy}}{\partial x}+\frac{a_{22}}{a_{11}}\ \frac{16f^4}{a}\ y N_{11}$$

$$-\frac{8\sqrt{a_{11}}\,f^3}{a}\ Q_y+\frac{8\sqrt{a_{22}}\ a_{12}f^3}{a\ a_{11}}\ Q_x+\sqrt{\frac{a}{a_{11}}}\ p_y=0, \qquad 4.4.2.18$$

$$\frac{1}{\sqrt{a_{11}}}\ \frac{\partial Q_1}{\partial x}+\frac{1}{\sqrt{a_{22}}}\ \frac{\partial Q_y}{\partial y}+\frac{xQ_x}{\sqrt{a_{11}}}\ \frac{16f^4}{a}\left[1+\frac{a}{4f^2a_{11}}\right]$$

$$+\frac{yQ_y}{\sqrt{a_{22}}}\ \frac{16f^4}{a}\left[1-\frac{a}{4f^2a_{22}}\right]+\frac{8f^3}{a}\left[N_{11}\sqrt{\frac{a_{22}}{a_{11}}}+N_{22}\sqrt{\frac{a_{11}}{a_{22}}}\right]+p_\zeta=0, \qquad 4.4.2.19$$

$$\frac{\partial M_x}{\partial x}+\left[\frac{16f^4}{a}-\frac{4f^2}{a_{11}}\right] xM_x+\sqrt{\frac{a_{11}}{a_{22}}}\ \frac{\partial M_{xy}}{\partial y}+\frac{a_{11}}{a_{22}}\ \frac{16f^4}{a}\ xM_y-\sqrt{\frac{a}{a_{22}}}\ Q_x=0, \qquad 4.4.2.20$$

$$\frac{\partial M_y}{\partial y}+\left[\frac{16f^4}{a}-\frac{4f^2}{a_{22}}\right] xM_y+\sqrt{\frac{a_{22}}{a_{11}}}\ \frac{\partial M_{xy}}{\partial x}+\frac{a_{22}}{a_{11}}\ \frac{16f^4}{a}\ yM_x$$

$$-\sqrt{\frac{a}{a^{11}}}\ Q_y=0. \qquad 4.4.2.21$$

V THE STRESS-STRAIN RELATIONS

5.1 GENERAL RELATIONS FOR ISOTROPIC MATERIALS

The stress-strain relations for an isotropic material and general curvilinear coordinates are given by the tensor equation

$$\tau^{mn} = \mu\left\{ g^{mk} g^{n\ell} + g^{m\ell} g^{nk} + \frac{2\nu}{1-2\nu} g^{mn} g^{k\ell} \right\} \gamma_{k\ell} \quad 5.1.1$$

where

$$\mu = \frac{E}{2(1+\nu)} \quad = \text{Shear Modulus,} \quad 5.1.2$$

$$E = \text{Young's Modulus,} \quad 5.1.3$$

$$\nu = \text{Poisson's Ratio.} \quad 5.1.4$$

In the case of a thin shell, the Kirchhoff assumption leads to the vanishing of $\gamma_{33}$ and to the neglect of $\tau^{33}$. Also, the assumption of negligible transverse shear strain voids the use of two of the six stress-strain relations. All these factors in conjunction with the special properties of the shell coordinate system reduces equation 5.1.1

to the following:

$$\tau^{\alpha\beta} = \Omega^{\alpha\beta\gamma\tau}\,\gamma_{\gamma\tau} \qquad 5.1.5$$

where the elastic constants $\Omega^{\alpha\beta\gamma\tau}$ are given by

$$\Omega^{\alpha\beta\gamma\tau} = \mu\left\{a^{\alpha\gamma}a^{\beta\tau} + a^{\alpha\tau}a^{\beta\gamma} + \frac{2\nu}{1-\nu}a^{\alpha\beta}a^{\gamma\tau}\right\} \qquad 5.1.6$$

If equation 5.1.5 is substituted into the definitions for the force and moment resultants (see equations 5.2.21 - 5.2.28), there will be obtained

$$\ell^{\alpha\beta} = h\Omega^{\alpha\beta\gamma\tau}\,\gamma^{o}_{\gamma\tau} \qquad 5.1.7$$

$$m^{\alpha\beta} = \frac{h^3}{12}\,\Omega^{\alpha\beta\gamma\tau}\,k_{\gamma\tau} \qquad 5.1.8$$

In these latter two equations, the strain tensor $\gamma_{\gamma\tau}$ has been expressed in terms of the strain tensor of the middle surface and the strain-curvature of the middle surface (see equation 3.3.19). The stress-strain relations in explicit form are listed as follows:

$$\ell^{11} = \frac{Eh}{1-\nu^2}\left\{a^{11}a^{11}\gamma^{o}_{11} + a^{12}a^{11}\gamma^{o}_{12} + \left[(1-\nu)a^{12}a^{12} + \nu a^{11}a^{22}\right]\gamma^{o}_{22}\right\}, \qquad 5.1.9$$

$$\ell^{12} = \frac{Eh}{1-\nu^2}\left\{a^{12}a^{11}\gamma^{o}_{11} + \left[\frac{1-\nu}{2}a^{11}a^{22} + \frac{1+\nu}{2}a^{12}a^{12}\right]\gamma^{o}_{12} + a^{12}a^{22}\gamma^{o}_{22}\right\}, \qquad 5.1.10$$

$$\ell^{22} = \frac{Eh}{1-\nu^2}\left\{\left[(1-\nu)a^{12}a^{12} + \nu a^{11}a^{22}\right]\gamma^{o}_{11} + a^{12}a^{22}\gamma^{o}_{12} + a^{22}a^{22}\gamma^{o}_{22}\right\}, \qquad 5.1.11$$

$$m^{11} = \frac{Eh^3}{12(1-\nu^2)}\left\{a^{11}a^{11}k_{11} + a^{12}a^{11}k_{12} + \left[(1-\nu)a^{12}a^{12} + \nu a^{11}a^{22}\right]k_{22}\right\}, \qquad 5.1.12$$

$$m^{12} = \frac{Eh^3}{12(1-\nu^2)}\left\{a^{12}a^{11}k_{11} + \left[\frac{1-\nu}{2}a^{11}a^{22} + \frac{1+\nu}{2}a^{12}a^{12}\right]k_{12} + a^{12}a^{22}k_{22}\right\}, \qquad 5.1.13$$

$$m^{22} = \frac{Eh^3}{12(1-\nu^2)}\left\{\left[(1-\nu)a^{12}a^{12} + \nu a^{11}a^{22}\right]k_{11} + a^{12}a^{22}k_{12} + a^{22}a^{22}k_{22}\right\}. \qquad 5.1.14$$

## 5.2 POLAR PARAMETERS

The polar parameters, r and $\theta$, being orthogonal, yield stress-strain relations which are much simplified. In tensor form these are as follows:

$$\ell^{11} = \frac{Eh}{1-\nu^2}\left\{a^{11}a^{11}\gamma^{\circ}_{11} + \nu\, a^{11}a^{22}\gamma^{\circ}_{22}\right\}, \qquad 5.2.1$$

$$\ell^{12} = \frac{Eh}{1-\nu^2}\left\{\frac{1-\nu}{2}\, a^{11}a^{22}\gamma^{\circ}_{12}\right\}, \qquad 5.2.2$$

$$\ell^{22} = \frac{Eh}{1-\nu^2}\left\{a^{22}a^{22}\gamma^{\circ}_{22} + \nu\, a^{11}a^{22}\gamma^{\circ}_{11}\right\} \qquad 5.2.3$$

$$m^{11} = \frac{Eh^3}{12(1-\nu^2)}\left\{a^{11}a^{11}k_{11} + \nu a^{11}a^{22}k_{22}\right\} \qquad 5.2.4$$

$$m^{12} = \frac{Eh^3}{12(1-\nu^2)}\left\{\frac{1-\nu}{2}\, a^{11}a^{22}k_{12}\right\}, \qquad 5.2.5$$

$$m^{22}\,\frac{Eh^3}{12(1-\nu^2)}\left\{a^{22}a^{22}k_{22} + \nu a^{11}a^{22}k_{11}\right\}. \qquad 5.2.6$$

In these relations, the components of the contravariant metric tensor are given by equations 2.1.21, 2.1.22, and 2.1.23. The stress-strain relations, in the case of the orthogonal coordinates, assume especially simple forms if expressed in terms of physical components.

$$N_r = \frac{Eh}{1-\nu^2}\left(\epsilon_r^o + \nu\,\epsilon_\theta^o\right), \tag{5.2.7}$$

$$N_{r\theta} = \frac{Eh}{1-\nu^2}\left(1-\nu\right)\epsilon_{r\theta}^o, \tag{5.2.8}$$

$$N_\theta = \frac{Eh}{1-\nu^2}\left(\epsilon_\theta^o + \nu\epsilon_r^o\right), \tag{5.2.9}$$

$$M_r = \frac{Eh}{12\left(1-\nu^2\right)}\left(K_r + \nu K_\theta\right), \tag{5.2.10}$$

$$M_{r\theta} = \frac{Eh^3}{12\left(1-\nu^2\right)}\left(1-\nu\right)K_{r\theta}, \tag{5.2.11}$$

$$M_\theta = \frac{Eh^3}{12\left(1-\nu^2\right)}\left(K_\theta + \nu K_r\right). \tag{5.2.12}$$

## 5.3 CARTESIAN PARAMETERS

The stress-strain relations to be used with the Cartesian parameters take on rather imposing appearances.

$$\ell^{11} = \frac{Eh}{1-\nu^2} \cdot \frac{1}{16f^4(1+x^2+y^2)^2} \Big\{ (1+y^2)^2 \gamma^0_{11} - xy(1+y^2)\gamma^0_{12}$$

$$+ \left[ (1-\nu)x^2y^2 + \nu(1+x^2)(1+y^2) \right] \gamma^0_{22} \Big\},$$

5.3.1

$$\ell^{12} = \frac{Eh}{1-\nu^2} \cdot \frac{1}{16f^4(1+x^2+y^2)} \Big\{ -(xy)(1+y^2)\gamma^0_{11} + \left[ \left(\frac{1-\nu}{2}\right)(1+x^2)(1+y^2) + \left(\frac{1+\nu}{2}\right)x^2y^2 \right] \gamma^0_{12}$$

$$-(xy)(1+x^2)\gamma^0_{22} \Big\},$$

5.3.2

$$\ell_{22} = \frac{Eh}{1-\nu^2} \cdot \frac{1}{16f^4(1+x^2+y^2)} \Big\{ \left[ (1-\nu)x^2y^2 + \nu(1+y^2)(1+x^2) \right] \gamma^0_{11}$$

$$-(xy)(1+x^2)\gamma^0_{12} + (1+x^2)^2 \gamma^0_{22} \Big\},$$

5.3.3

$$m^{11} = \frac{Eh^3}{12(1-\nu^2)} \cdot \frac{1}{16f^4(1+x^2+y^2)} \Big\{ (1+y^2)^2 k_{11} - xy(1+y^2) k_{12}$$

$$+ \left[ (1-\nu)x^2y^2 + \nu(1+x^2)(1+y^2) \right] k_{22} \Big\},$$

5.3.4

$$m^{12} = \frac{Eh^3}{12(1-\nu^2)} \cdot \frac{1}{16f^4(1+x^2+y^2} \Big\{ -(xy)(1+y^2) k_{11}$$

$$+ \left[ \left(\frac{1-\nu}{2}\right)(1+x^2)(1+y^2) + \left(\frac{1+\nu}{2}\right)x^2y^2 \right] k_{12} - (xy)(1+x^2) k_{22} \Big\},$$

5.3.5

$$m_{22} = \frac{Eh^3}{12(1-\nu^2)} \cdot \frac{1}{16f^4(1+x^2+y^2)} \Big\{ \big[ (1-\nu)x^2y^2 + \nu(1+x^2)(1+y^2) \big] k_{11} - (xy)(1+x^2)\, k_{12} + (1+x^2)^2\, k_{22} \Big\}. \tag{5.3.6}$$

These equations look just as, if not more imposing, when the physical components of the tensors are used.

$$N_x = \left[\frac{Eh}{1-\nu^2}\right]\left[1 + \frac{x^2y^2}{1+x^2+y^2}\right]^{3/2} \Big\{ \epsilon_x^{\circ} - \frac{xy}{\sqrt{(1+x^2)(1+y^2)}}\, \epsilon_{xy}^{\circ} + \left[1 + \frac{(1-\nu)}{\nu}\, \frac{x^2y^2}{(1+x^2)(1+y^2)}\right] \nu\, \epsilon_y^{\circ} \Big\}, \tag{5.3.7}$$

$$N_{xy} = \left[\frac{Eh}{1-\nu^2}\right]\left[1 + \frac{x^2y^2}{1+x^2+y^2}\right]^{3/2} \Big\{ \left[1 + \left(\frac{1+\nu}{1-\nu}\right) \frac{x^2y^2}{(1+x^2)(1+y^2)}\right] \epsilon_{xy}^{\circ} - \left[\left(\frac{2}{1-\nu}\right) \frac{xy}{\sqrt{(1+x^2)(1+y^2)}}\right]\left[\epsilon_x^{\circ} + \epsilon_y^{\circ}\right] \Big\}, \tag{5.3.8}$$

$$N_y = \left[\frac{Eh}{1-\nu^2}\right]\left[1 + \frac{x^2y^2}{1+x^2+y^2}\right]^{3/2} \Big\{ \epsilon_y^{\circ} - \frac{xy}{\sqrt{1+x^2+y^2}}\, \epsilon_{xy}^{\circ} + \left[1 + \frac{(1-\nu)}{\nu}\, \frac{x^2y^2}{(1+x^2)(1+y^2)}\right] \nu\, \epsilon_x^{\circ} \Big\}, \tag{5.3.9}$$

$$M_x = \left[\frac{Eh^3}{12(1-\nu^2)}\right]\left[1+\frac{x^2y^2}{1+x^2+y^2}\right]^{3/2}\left\{K_x - \frac{xy}{\sqrt{1+x^2+y^2}}\,K_{xy} + \left[1+\frac{(1-\nu)}{\nu}\,\frac{x^2y^2}{(1+x^2+y^2)}\right]\nu K_y\right\} \tag{5.3.10}$$

$$M_{xy} = \left[\frac{Eh^3}{12(1-\nu^2)}\right]\left[1+\frac{x^2y^2}{1+x^2+y^2}\right]^{3/2}\left\{\left[1+\frac{(1+\nu)}{(1-\nu)}\,\frac{x^2y^2}{(1+x^2)(1+y^2)}\right]K_{xy} - \left[\left(\frac{2}{1-\nu}\right)\frac{xy}{\sqrt{(1+x^2)(1+y^2)}}\right]\left[\epsilon_x^0+\epsilon_y^0\right]\right\}. \tag{5.3.11}$$

$$M_y = \left[\frac{Eh^3}{12(1-\nu^2)}\right]\left[1+\frac{x^2y^2}{1+x^2+y^2}\right]^{3/2}\left\{K_y - \frac{xy}{\sqrt{1+x^2+y^2}}\,K_{xy} + \left[1+\frac{(1-\nu)}{\nu}\,\frac{x^2y^2}{(1+x^2)(1+y^2)}\right]\nu K_x\right\}. \tag{5.3.12}$$

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# ERRATA SHEET

For Report 71G-1

Distortions and Stresses of
Paraboloidal Surface Structures, Part 1

7 February 1962

The authors have detected the following errors in Report 71G-1 (F. Y. Wan, Prof. J. Mar, "Distortions and Stresses of Paraboloidal Surface Structures, Part 1," 7 February 1962).

Kindly insert these pages into your copy of this report.

Page 2.12 Equation 2.1.43 should be

$$\frac{\partial \bar{n}}{\partial \gamma} = - \frac{1}{R_1} \bar{a}_1 = - \frac{\bar{a}_1}{2f\left[1+(\gamma)^2\right]^{3/2}}$$

Page 2.12 Equation 2.1.44 should be

$$\frac{\partial \bar{n}}{\partial \theta} = - \frac{1}{R_2} \bar{a}_2 = - \frac{\bar{a}_2}{2f\left[1+(\gamma)^2\right]^{1/2}}$$

Page 2.28 Equation 2.4.4 should be

$$\bar{g}_3 = \frac{\partial \bar{r}}{\partial \zeta} = \bar{n}$$

Page 3.1 Equation 3.1.2 should be

$$(dS)^2 = G_{mn} \, dx^m \, dx^n$$

Page 3.3 Equation 3.1.7 should be

$$\gamma_{mn} = \frac{1}{2} \left( \bar{g}_m \cdot \frac{\partial \bar{v}}{\partial x^m} + \bar{g}_n \cdot \frac{\partial \bar{v}}{\partial x^m} + \frac{\partial \bar{v}}{\partial x^m} \cdot \frac{\partial \bar{v}}{\partial x^n} \right)$$

**Page** 3.5 Equation 3.2.6 should be

$$\gamma_{22} = \left(v^2_{,2} - \frac{v^3}{R_2}\right) \left(1 - \frac{\zeta}{R_2}\right) a_{22}$$
$$+\frac{1}{2}\left\{\left(v^1_{,2}\ v^1_{,2}\right) a_{11} + \left(v^2_{,2} - \frac{v^3}{R_2}\right)^2 a_{22} + \left(\frac{a_{22} v^2}{R_2} + \frac{\partial v^3}{\partial \xi^2}\right)^2\right\}$$

**Page** 3.9 Equation 3.3.9 should be

$$\gamma_{11} = \left(u^1_{,1} - b^1_1 w\right) a_{11} + \left(u^2_{,1} - b^2_1 w\right) a_{12} + \zeta\left(\omega^1_{,1}\, a_{11} + \omega^2_{,1}\, a_{12}\right)$$

**Page** 3.9 Equation 3.3.12 should be

$$\gamma_{23} = \frac{1}{2}\left\{\omega^1 a_{12} + \omega^2 a_{22} + u^1 b_{12} + u^2 b_{22} + \frac{\partial w}{\partial \xi^2} + \zeta\left(\omega^1 b_{12} + \omega^2 b_{22}\right)\right\}$$

**Page** 3.11 Equation 3.3.17 should be

$$\omega_2 = -b_{12}\, u^1 - b_{22}\, u^2 - \frac{\partial w}{\partial \xi^2} .$$

**Page** 3.16 Equation 3.4.22 should be

$$\omega^\circ_r = -\frac{u^\circ_r}{2f\left[1+\gamma^2\right]^{3/2}} - \frac{1}{2f\sqrt{1+\gamma^2}} \frac{\partial w}{\partial \gamma}$$

**Page** 3.16 Equation 3.4.26 should be

$$K_{r\theta} = \frac{\sqrt{1+\gamma^2}}{4f\gamma} \frac{\partial \omega^\circ_r}{\partial \theta} + \frac{1}{4f} \frac{\partial \omega^\circ_\theta}{\partial \gamma} - \frac{\omega^\circ_\theta}{4f\gamma}$$

**Page** 3.17 Equation 3.5.2 should be

$$\gamma^\circ_{22} = \left(\frac{\partial u^2}{\partial y} + \frac{y u^2}{1+(x)^2+(y)^2}\right) 4f^2\left[1+(y)^2\right] + \left(\frac{\partial u^1}{\partial y} + \frac{x u^2}{1+(x)^2+(y)^2}\right) 4f^2 xy$$
$$- \frac{2f w}{\sqrt{1+(x)^2+(y)^2}}$$

Page 3.17 Equation 3.5.3 should be

$$\gamma_{12}^{o} = \frac{1}{2}\Big\{\Big(\frac{\partial u^1}{\partial y} + \frac{x u^2}{1+(x)^2+(y)^2}\Big) 4f^2[1+(x)^2] + \Big(\frac{\partial u^2}{\partial x} + \frac{y u^1}{1+(x)^2+(y)^2}\Big) 4f^2[1+(y)^2]$$

$$+ \Big(\frac{\partial u^1}{\partial x} + \frac{x u^1}{1+(x)^2+(y)^2} + \frac{\partial u^2}{\partial y} + \frac{y u^2}{1+(x)^2+(y)^2}\Big) 4f^2 x y - 2 b_{12} w\Big\}$$

Page 4.1 Line 2, Paragraph 2, $G_n$ should be $G_m$

**Page 4.6 Line 1, Paragraph 1,** $\ell^1$ **and** $\ell^2$ **should be** $\bar{\ell}^1$ **and** $\bar{\ell}^2$

Page 4.7 Equation 4.2.2 should be

$$\bar{\ell}^2 \sqrt{A_{11}} \; d\xi^1 = \int_{-\frac{h}{2}^*}^{\frac{h}{2}^*} \bar{\tau}^2 \sqrt{G_{11}} \; d\xi^1 \, d\zeta^*$$

Page 4.12 Equation 4.3.1 should be

$$\bar{L}^1 \, d\xi^2 = \bar{\ell}^1 \sqrt{A_{22}} \; d\xi^2$$

Page 4.18 Equation 4.4.1 should be

$$\ell^{\beta\alpha}_{,\beta} - q^{\beta} \, b^{\alpha}_{\beta} + F^{\alpha} = 0$$

Page 4.25 Equation 4.4.2.21 should be

$$\frac{\partial M_y}{\partial y} + \Big[\frac{16f^4}{a} - \frac{4f^2}{a_{22}}\Big] x M_y + \sqrt{\frac{a_{22}}{a_{11}}} \; \frac{\partial M_{xy}}{\partial x} + \frac{a_{22}}{a_{11}} \; \frac{16f^4}{a} \, y M_x$$

$$- \sqrt{\frac{a}{a^{11}}} \; Q_y = 0$$

Page 5.3 Equation 5.1.12 should be

$$m^{11} = \frac{Eh^3}{12(1-\nu^2)} \left\{ a^{11} a^{11} k_{11} + a^{12} a^{11} k_{12} + \left[ (1-\nu) a^{12} a^{12} + \nu a^{11} a^{22} \right] k_{22} \right\}$$

Page 5.4 Equation 5.2.6 should be

$$m^{22} = \frac{Eh^3}{12(1-\nu^2)} \left\{ a^{22} a^{22} k_{22} + \nu a^{11} a^{22} k_{11} \right\}$$

Page 5.5 Equation 5.2.10 should be

$$M_r \doteq \frac{Eh^3}{12(1-\nu^2)} \left( K_r + \nu K_\theta \right)$$

Page 5.6 Equation 5.3.5 should be

$$m^{12} = \frac{Eh^3}{12(1-\nu^2)} \cdot \frac{1}{16f^4 (1+x^2+y^2)} \left\{ -(xy)(1+y^2) k_{11} + \left[ \left(\frac{1-\nu}{2}\right)(1+x^2)(1+y^2) + \left(\frac{1+\nu}{2}\right) x^2 y^2 \right] k_{12} - (xy)(1+x^2) k_{22} \right\}$$

Page 5.7 Equation 5.3.6 should be

$$m_{22} = \frac{Eh^3}{12(1-\nu^2)} \cdot \frac{1}{16f^4 (1+x^2+y^2)} \left\{ \left[ (1-\nu) x^2 y^2 + \nu (1+x^2)(1+y^2) \right] k_{11} - (xy)(1+x^2) k_{12} + (1+x^2)^2 k_{22} \right\}$$

Page 5.7 Equation 5.3.8 should be

$$N_{xy} = \left[\frac{Eh}{1-\nu^2}\right] \left[ 1 + \frac{x^2 y^2}{1+x^2+y^2} \right]^{3/2} \left\{ \left[ 1 + \left(\frac{1+\nu}{1-\nu}\right) \frac{x^2 y^2}{(1+x^2)(1+y^2)} \right] \epsilon^{\circ}_{xy} - \left[ \left(\frac{2}{1-\nu}\right) \frac{xy}{\sqrt{(1+x^2)(1+y^2)}} \right] \left[ \epsilon^{\circ}_{x} + \epsilon^{\circ}_{y} \right] \right\}$$

23 March 1962

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